# Creation of Photonic Cat and GKP States Using Modulated Electrons - Reference Material

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# RM 1 Basic formalism of the interaction between comb electrons and a photonic mode

In this section, we derive the formalism for the interaction between a comb electron and a general state of a photonic mode, which is used throughout the main text. The initial joint state of a photonic mode $|\psi\rangle_{\mathrm{ph}}$ and an electron with a wavefunction in the energy representation $|\psi\rangle_{\mathrm{e}}$ is:

$$|\Psi_{\mathrm{in}}\rangle = |\psi\rangle_{\mathrm{e}} \otimes |\psi\rangle_{\mathrm{ph}}. \tag{1}$$

The interaction between a free electron and a photonic mode is captured by the scattering matrix $S$ (for more information on the assumptions of this model look at [1-4]):

$$S = D_{bg_{\mathrm{Q}}} = e^{g_{\mathrm{Q}} b a^\dagger - g_{\mathrm{Q}}^* b^\dagger a}, \tag{2}$$

where $g_{\mathrm{Q}}$ is the coupling constant between the photonic mode and free electrons; $a, a^\dagger$ are the annihilation and creation operators for the photonic mode; $b$, $b^\dagger$ are the operators describing the electron translation in energy, which correspond to the emission or absorption of a single photon; $D_x = \exp(xa^\dagger - x^* a)$ is a coherent displacement operator [5]. The commutation relations for the operators are:

$$[b, b^\dagger] = 0, \qquad [a, a^\dagger] = 1 \ . \tag{3}$$

Let us consider a comb electron (Dirac comb) with an energy difference $\hbar\omega$ (Fig. 2a in the main text):

$$|\mathrm{comb}(\varphi)\rangle_{\mathrm{e}} \propto \sum_{k=-\infty}^{\infty} e^{i\varphi k} |E_0 + \hbar\omega \cdot k\rangle_{\mathrm{e}}\,, \tag{4}$$

where $\varphi \in \mathbb{R}$ is a general phase. Such states are eigenfunctions of the electron energy translation operators (and the scattering matrix):

$$b|\mathrm{comb}(\varphi)\rangle_{\mathrm{e}} = e^{i\varphi}|\mathrm{comb}(\varphi)\rangle_{\mathrm{e}}, \tag{5}$$

thus proving the equality $S = D_{g_{\mathrm{Q}} b} \Leftrightarrow D_{g_{\mathrm{Q}} e^{i\varphi}}$ [6]. Using this relation, we derive the joint state after each interaction of a general photonic state with a comb electron:

$$|\Psi_{\mathrm{out}}\rangle = S|\Psi_{\mathrm{in}}\rangle = D_{g_{\mathrm{Q}} e^{i\varphi}} |\mathrm{comb}(\varphi)\rangle_{\mathrm{e}} \otimes |\psi_{\mathrm{i}}\rangle_{\mathrm{ph}}. \tag{6}$$

We can see that the scattering matrix acts as a displacement operator on the photonic mode and that the comb electron is an eigenfunction of the scattering matrix $S$. Therefore, the photonic state and the electron state are separable (i.e., not entangled) after the interaction. For the case of an initial photonic mode in a vacuum state, the final state will be a coherent state. It is convenient to introduce the following notation:

$$D_{g_{\mathrm{Q}} e^{i\varphi}} |0\rangle_{\mathrm{ph}} = \left|g_{\mathrm{Q}} e^{i\varphi}\right\rangle_{\mathrm{ph}}, \tag{7}$$

where $\left|g_{\mathrm{Q}} e^{i\varphi}\right\rangle_{\mathrm{ph}}$ is a coherent state with an amplitude $\left|g_{\mathrm{Q}}\right|$ and phase $\varphi$.

Eq. (7) can be generalized to describe emission resulting from any electron using Fourier decomposition, which makes it a very strong tool. Any electron can be decomposed to a series (or an integral for the continuous case) over comb electrons, and therefore the final photonic state can be written as a sum of coherent states on a circle of radius $g_{\mathrm{Q}}$:

$$|\psi\rangle_{\mathrm{e}} = \sum_{k=-N}^{N} c_k |\mathrm{comb}(\phi_k)\rangle_{\mathrm{e}}\,, \tag{8}$$

while $c_k = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} c_n e^{i\left(\frac{2\pi}{2N+1}k\right)n}$ and $\phi_k = \frac{2\pi}{2N+1}k$. Thus:

$$|\Psi_{\mathrm{out}}\rangle = S|\Psi_{\mathrm{in}}\rangle = S\left(|\psi\rangle_{\mathrm{ph}} \otimes \sum_k c_k\, |\mathrm{comb}(\phi_k)\rangle_{\mathrm{e}}\right) = \sum_k c_k D_{g_{\mathrm{Q}} e^{i\phi_k}} |\psi\rangle_{\mathrm{ph}} \otimes |\mathrm{comb}(\phi_k)\rangle_{\mathrm{e}}. \tag{9}$$

Eq. (9) gives a general expression for the interaction between an arbitrary photonic state and an arbitrarily shaped electron.

# RM 2 Creation of cat states

## RM 2.1 Creation of 2-component cat state

Let us introduce a comb electron with an energy difference of $N\hbar\omega$ and a shift in energy by $m\hbar\omega$:

$$|\mathrm{comb}_N^m\rangle_{\mathrm{e}} \propto \sum_{k=-\infty}^{\infty} |E_0 + \hbar\omega \cdot (Nk - m)\rangle_{\mathrm{e}} \tag{10}$$

To create a 2-component cat state, we consider a comb electron with an energy difference of $2\hbar\omega$:

$$|\mathrm{comb}_{\mathrm{even}}\rangle_{\mathrm{e}} = |\mathrm{comb}_2^0\rangle_{\mathrm{e}} \propto \sum_{k=-\infty}^{\infty} |E_0 + \hbar\omega \cdot 2k\rangle_{\mathrm{e}}\,. \tag{11}$$

We have two options for the comb electron $|\mathrm{comb}_{N=2}^{\mathrm{m}}\rangle_{\mathrm{e}}$, as illustrated in Fig. 2c in the main text. The first one is an even comb, as in Eq. (11), and the second one is an odd comb electron:

$$|\mathrm{comb}_{\mathrm{odd}}\rangle_{\mathrm{e}} = |\mathrm{comb}_2^1\rangle_{\mathrm{e}} \propto \sum_{k=-\infty}^{\infty} |E_0 + \hbar\omega \cdot (2k-1)\rangle_{\mathrm{e}}\,. \tag{12}$$

To create a cat state, the photonic mode is initially prepared in the vacuum state. Here we show the explicit calculation for an even comb (starting with the odd comb is equivalent since $E_0$ is defined arbitrarily). Thus, we consider the following initial joint-state:

$$|\Psi_{\mathrm{in}}\rangle = |\mathrm{comb}_{\mathrm{even}}\rangle_{\mathrm{e}} \otimes |0\rangle_{\mathrm{ph}}. \tag{13}$$

In order to calculate $S|\Psi_{\mathrm{in}}\rangle$, the final state after the interaction, it is convenient to decompose the even comb electron into the basis of comb electrons with $\hbar\omega$ energy differences and different phases (i.e., use the Discrete Fourier Transform according to Eq. (8)). Firstly, we use the general comb electron state defined in Eq. (4). Then, we decompose the even and odd combs in the following way:

$$\begin{cases} |\mathrm{comb_{even}}\rangle_\mathrm{e} = (|\mathrm{comb}(0)\rangle_\mathrm{e} + |\mathrm{comb}(\pi)\rangle_\mathrm{e})/\sqrt{2} \\ |\mathrm{comb_{odd}}\rangle_\mathrm{e} = (|\mathrm{comb}(0)\rangle_\mathrm{e} - |\mathrm{comb}(\pi)\rangle_\mathrm{e})/\sqrt{2} \end{cases}. \tag{14}$$

Alternatively:

$$\Leftrightarrow \begin{cases} |\mathrm{comb}(0)\rangle_\mathrm{e} = (|\mathrm{comb_{even}}\rangle_\mathrm{e} + |\mathrm{comb_{odd}}\rangle_\mathrm{e})/\sqrt{2} \\ |\mathrm{comb}(\pi)\rangle_\mathrm{e} = (|\mathrm{comb_{even}}\rangle_\mathrm{e} - |\mathrm{comb_{odd}}\rangle_\mathrm{e})/\sqrt{2} \end{cases}. \tag{15}$$

In this new basis, the final joint state after interaction of an even comb electron with a general photonic state is calculated by using Eqs. (6,7):

$$|\Psi_\mathrm{out}\rangle = S|\Psi_\mathrm{in}\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{comb}(0)\rangle_\mathrm{e} \otimes |g_\mathrm{Q}\rangle_\mathrm{ph} + |\mathrm{comb}(\pi)\rangle_\mathrm{e} \otimes |-g_\mathrm{Q}\rangle_\mathrm{ph}\right) =$$
$$= \frac{1}{2}|\mathrm{comb_{even}}\rangle_\mathrm{e} \otimes \left(|g_\mathrm{Q}\rangle_\mathrm{ph} + |-g_\mathrm{Q}\rangle_\mathrm{ph}\right) + \frac{1}{2}|\mathrm{comb_{odd}}\rangle_\mathrm{e} \otimes \left(|g_\mathrm{Q}\rangle_\mathrm{ph} - |-g_\mathrm{Q}\rangle_\mathrm{ph}\right). \tag{16}$$

The definition of even and odd cat states is [7]:

$$|\mathrm{cat_{even/odd}}\rangle_\mathrm{ph} \propto \frac{1}{2}\left(|g_\mathrm{Q}\rangle_\mathrm{ph} \pm |-g_\mathrm{Q}\rangle_\mathrm{ph}\right), \tag{17}$$

with the plus (minus) sign referring to the even (odd) comb electron.

We notice that in order to get a cat state in the photonic mode, an electron-energy post-selection is necessary. If the electron is traced out, then the photonic mode is left in a mixed cat state:

$$\rho_\mathrm{ph} = \frac{1}{2}\left(|\mathrm{cat_{even}}\rangle_\mathrm{ph}\langle\mathrm{cat_{even}}|_\mathrm{ph} + |\mathrm{cat_{odd}}\rangle_\mathrm{ph}\langle\mathrm{cat_{odd}}|_\mathrm{ph}\right). \tag{18}$$

However, if we post-select the electron with specific energy, then:

$$|\psi_\mathrm{f}\rangle_\mathrm{ph} = \begin{cases} |\mathrm{cat_{even}}\rangle_\mathrm{ph}, \text{if the energy is even,} \\ |\mathrm{cat_{odd}}\rangle_\mathrm{ph}, \text{if the energy is odd.} \end{cases} \tag{19}$$

In such a way, the electron holds information about the parity of the photons in the photonic mode. Therefore, by electron energy post-selection, cat states can be generated in the photonic mode in a heralded process.

## RM 2.2 Creation of N-component cat states

After interaction of a $|\mathrm{comb_N^0}\rangle_\mathrm{e}$ with a vacuum photonic state, the final state is:

$$|\Psi_\mathrm{out}\rangle = |\mathrm{comb_N^{-\mathit{k}}}\rangle_\mathrm{e} \otimes \sum_{k=0}^{N-1}\left(e^{\frac{-|g_Q|^2}{2}}\sum_n \frac{(g_\mathrm{Q})^{Nn+k}}{\sqrt{(Nn+k)!}}|Nn+k\rangle_\mathrm{ph}^\mathrm{Fock}\right), \tag{20}$$

where the $N$ factor appears since this electron can be written as a superposition of $N$ comb electrons with different phases. $|N\rangle_\mathrm{ph}^\mathrm{Fock}$ denotes the $N^\mathrm{th}$ Fock number state. We identify the photonic state as the $N$-component cat state $|\mathrm{cat}_N^k\rangle_\mathrm{ph}$:

$$\left|\text{cat}_N^k\right\rangle_{\text{ph}} \equiv \frac{1}{c_N^k}\sum_{m=0}^{N-1} e^{-i2\pi\frac{km}{N}}\left|e^{i2\pi\frac{m}{N}}g_{\text{Q}}\right\rangle_{\text{ph}} = \frac{N}{c_N^k}e^{\frac{-|g_{\text{Q}}|^2}{2}}\sum_n \frac{\left(g_{\text{Q}}\right)^{Nn+k}}{\sqrt{(Nn+k)!}}|Nn+k\rangle_{\text{ph}}^{\text{Fock}}, \quad (21)$$

where this set of $N$ states is called the $N$-component cat state set [8], defined as $\left|\text{cat}_N^k\right\rangle_{\text{ph}}$. The index $k$ defines the particular state from Eq. (19). $c_N^k$ is the normalization factor of the $N$-componen cat states (Eq. (21)). Thus, finally, we get:

$$|\Psi_{\text{out}}\rangle = \frac{1}{N}\sum_{k=0}^{N-1} c_N^k\left|\text{comb}_{\text{N}}^{m-k}\right\rangle_{\text{e}} \otimes \left|\text{cat}_N^k\right\rangle_{\text{ph}}. \quad (22)$$

Hence, if we want to obtain $|\text{cat}_N^k\rangle$, we need to measure the energy $E_0 + \hbar\omega\cdot(Nm-k)$, where $m$ is an integer. In the limit of a mono-energetic electron ($N \gg 1$), the electron energies are entangled to photonic Fock-states [3, 6].

### RM 2.3 Post-selection probability for cat states

The probability of post-selecting the different cat states is an important property. Let us first calculate the probabilities to post-select a two-component cat state. From Eq. (16), one can notice that the probability of post-selecting even energy is $P(\text{even}) = \left\|\left(\left|g_{\text{Q}}\right\rangle_{\text{ph}} + \left|-g_{\text{Q}}\right\rangle_{\text{ph}}\right)/2\right\|^2 = \left(1 + e^{-2|g_{\text{Q}}|^2}\right)/2$ and the probability of post-selecting odd energy is $P(\text{odd}) = \left\|\left(\left|g_{\text{Q}}\right\rangle_{\text{ph}} - \left|-g_{\text{Q}}\right\rangle_{\text{ph}}\right)/2\right\|^2 = \left(1 - e^{-2|g_{\text{Q}}|^2}\right)/2$. According to these equations, for a small coupling constant $\left|g_{\text{Q}}\right| \ll 1$, the probability to get an even cat state will be ~1 while the displacements will be ~0.

Following Eq. (22), the probability of post-selecting a specific N-component cat state $\left|\text{cat}_N^k\right\rangle$ is:

$$P_N^k = \frac{1}{N^2}\left|c_N^k\right|^2 = \frac{1}{N^2}\left\|\sum_{m=0}^{N-1} e^{-i2\pi\frac{km}{N}}\left|e^{i2\pi\frac{m}{N}}g_{\text{Q}}\right\rangle_{\text{ph}}\right\|^2. \quad (23)$$

We can see that for all $N$, the probability $P_N^k$ is maximal for $k=0$. Moreover $P_N^k$ goes down monotonically with k.

## RM 3 1D and 2D grids of coherent states

Here we propose a protocol for preparing 1D and 2D grids of coherent states using multiple comb electrons interacting with the same photonic mode.

### RM 3.1 Creating a 1D grid of coherent states- final state analytical expression

After an interaction with an even comb electron, the resulting joint state is:

$$|\Psi_{\text{out}}\rangle = \frac{1}{2}|\text{comb}_{\text{even}}\rangle_{\text{e}} \otimes \left(D_{g_{\text{Q}}} + D_{-g_{\text{Q}}}\right)|\psi_{\text{i}}\rangle_{\text{ph}} + \frac{1}{2}|\text{comb}_{\text{odd}}\rangle_{\text{e}} \otimes \left(D_{g_{\text{Q}}} - D_{-g_{\text{Q}}}\right)|\psi_{\text{i}}\rangle_{\text{ph}}, \quad (24)$$

Thus, after post-selecting even electron energy, the final photonic state is:

$$|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}} \propto \left(D_{g_{\mathrm{Q}}} + D_{-g_{\mathrm{Q}}}\right)|\psi_{\mathrm{i}}\rangle_{\mathrm{ph}}. \quad (25)$$

After post-selecting odd electron energy, the photonic state is:

$$|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}} \propto \left(D_{g_{\mathrm{Q}}} - D_{-g_{\mathrm{Q}}}\right)|\psi_{\mathrm{i}}\rangle_{\mathrm{ph}}. \quad (26)$$

Therefore, to calculate the photonic state after multiple even comb-electron interactions, we need to multiply the photonic state by the sum (i.e., Eq. (25)) or difference (i.e., Eq. (26)) of the displacement operators, depending on the post-selected energy. For the case of an initial odd comb electron, the photonic states that are coupled to the electron's state will be replaced with each other (up to a global phase), replacing Eqs. (25, 26). Using Eqs. (25, 26), we derive the final photonic state after many interactions with even comb electrons.

After $m+n$ cycles of comb electron-photon interaction with a coupling constant $g_{\mathrm{Q}}$, and provided that we measured (post-selected) $m$ times even energies and $n$ times odd energies, the resulting photonic state is:

$$|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}} \propto \left(D_{g_{\mathrm{Q}}} + D_{-g_{\mathrm{Q}}}\right)^{m}\left(D_{g_{\mathrm{Q}}} - D_{-g_{\mathrm{Q}}}\right)^{n}|\psi_{\mathrm{i}}\rangle_{\mathrm{ph}}. \quad (27)$$

Using the displacement operator's property $D_{\alpha}D_{\beta} = e^{\frac{1}{2}(\alpha\beta^{*}-\alpha^{*}\beta)}D_{\alpha+\beta}$, one can notice that the order of post-selections does not change the final state. Using the binomial theorem, Eq. (27) can be further simplified:

$$|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}} = \left|N_{\psi}\right|^{-1/2}\sum_{\alpha=0}^{m}\sum_{\beta=0}^{n}\binom{m}{\alpha}\binom{n}{\beta}(-1)^{\beta}D_{g_{\mathrm{Q}}(m+n-2\beta-2\alpha)}|\psi_{\mathrm{i}}\rangle_{\mathrm{ph}}\,, \quad (28)$$

where $\left|N_{\psi}\right|$ is a normalization factor and $\binom{m}{\alpha} = \frac{m!}{\alpha!(m-\alpha)!}$ are the binomial coefficients.

## RM 3.2 Creating a 2D coherent grid - final state analytical expression

In general, the coupling constant has a phase, which can be controlled by controlling the phase of the shaping laser in the scheme presented in Fig. 1a in the main text. A general coupling constant can be written as $g_{\mathrm{Q}} = e^{i\cdot\arg(g_{\mathrm{Q}})}|g_{\mathrm{Q}}|$. To generate a 2D coherent grid, we use even combs with different relative phases. Let us calculate the simplest case, a rectangular grid with two orthogonal coupling constants $g_{\mathrm{Q1}} = i|g_{\mathrm{Q1}}|$ and $g_{\mathrm{Q2}} = |g_{\mathrm{Q2}}|$. First, we create a 1D coherent grid by interactions with $i\left|g_{\mathrm{Q1}}\right|$, and afterward, interactions of coupling constant $\left|g_{\mathrm{Q2}}\right|$ to expand the 1D grid to a 2D grid.

We consider $k+l$ cycles of comb electron-photon interaction with coupling constant $i\left|g_{Q1}\right|$, followed by measurements post-selecting even/odd energies $k/l$ times, respectively. Afterward, we consider $m+n$ cycles of comb electron-photon interaction with coupling constant $\left|g_{\mathrm{Q2}}\right|$, followed by measurements post-selecting even/odd energies $m/n$ times, respectively. The resulting final photonic state is calculated using the approach presented in RM 3.1:

$$|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}} \propto \left(D_{|g_{\mathrm{Q2}}|} - D_{-|g_{\mathrm{Q2}}|}\right)^{n}\left(D_{|g_{\mathrm{Q2}}|} + D_{-|g_{\mathrm{Q2}}|}\right)^{m}\left(D_{i|g_{\mathrm{Q1}}|} - D_{-i|g_{\mathrm{Q1}}|}\right)^{l}\left(D_{i|g_{\mathrm{Q1}}|} + D_{-i|g_{\mathrm{Q1}}|}\right)^{k}|0\rangle_{\mathrm{ph}}. \quad (29)$$

Using the binomial theorem and Eq. (28), the final photonic state is:

$$\left|N_\psi\right|^{-1/2}\sum_{\alpha=0}^{m}\sum_{\beta=0}^{n}\sum_{\gamma=0}^{k}\sum_{\delta=0}^{l}\binom{m}{\alpha}\binom{n}{\beta}\binom{k}{\gamma}\binom{l}{\delta}(-1)^{\beta+\delta}D_{|g_{\mathrm{Q2}}|(2\alpha+2\beta-m-n)}D_{i|g_{\mathrm{Q1}}|(2\delta+2\gamma-k-l)}\,|0\rangle_{\mathrm{ph}}, \quad (30)$$

This equation represents a rectangular grid of coherent states. For example, if all the post-selected energies are even, i.e., $n = l = 0$, then the state is a rectangular grid state with positive coefficients on the complex plane. The grid spacing in the imaginary direction is $2\left|g_{\mathrm{Q1}}\right|$ and in the real direction is $2\left|g_{Q2}\right|$. Choosing odd energies will add local phases to the coefficients of the coherent states, from the element $(-1)^{\beta+\delta}$.

To create a general grid of coherent states (not only a rectangle shape) one should use general complex coupling constants $\tilde{g}_{\mathrm{Q1}}, \tilde{g}_{\mathrm{Q2}}$. The amplitudes of $\tilde{g}_{\mathrm{Q1}}, \tilde{g}_{\mathrm{Q2}}$ define the distances between the points in each axis, and their phases define the angles of the grid. For example, in Eq. (30), the relative phase of $g_{\mathrm{Q1}}$ and $g_{\mathrm{Q2}}$ is $\frac{\pi}{2}$, which creates a rectangular grid. For the general case:

$$|\psi\rangle_{\mathrm{ph}} \propto \sum_{\alpha=0}^{m}\sum_{\beta=0}^{n}\sum_{\gamma=0}^{k}\sum_{\delta=0}^{l}\binom{m}{\alpha}\binom{n}{\beta}\binom{k}{\gamma}\binom{l}{\delta}(-1)^{\beta+\delta}D_{\tilde{g}_{\mathrm{Q2}}(2\alpha+2\beta-m-n)}D_{\tilde{g}_{\mathrm{Q1}}(2\delta+2\gamma-k-l)}\,|0\rangle. \quad (31)$$

In order to examine the state in Eq. (31), we further express it in terms of the $x$ and $p$ quadratures. We use the following formulas, connecting the displacement operator to the transition in position and momentum operators [9]:

$$D_c = e^{-i\sqrt{2}c\hat{p}}, \qquad D_{ic} = e^{i\sqrt{2}c\hat{x}}, \quad (32)$$

where $c$ is a real parameter. Using these relations, the final state can be calculated in the coordinate and momentum representation. Let us use Eq. (32) to calculate the quadrature representation in $x$ and $p$ coordinates representations of Eq. (31):

$$\langle x|\psi\rangle_{\mathrm{ph}} \propto \sum_{\alpha,\beta,\gamma,\delta}\binom{m}{\alpha}\binom{n}{\beta}\binom{k}{\gamma}\binom{l}{\delta}(-1)^{\beta+\delta}e^{i\left(B(A+C)+C(B+D)\right)}e^{i\sqrt{2}(B+D)\left(x-\sqrt{2}(A+C)\right)}e^{-\frac{1}{2}\left(x-\sqrt{2}(A+C)\right)^2}, \quad (33)$$

$$\langle p|\psi\rangle_{\mathrm{ph}} \propto \sum_{\alpha,\beta,\gamma,\delta}\binom{m}{\alpha}\binom{n}{\beta}\binom{k}{\gamma}\binom{l}{\delta}(-1)^{\beta+\delta}e^{i\left(B(A+C)+C(B+D)\right)}e^{-i\sqrt{2}(A+C)p}e^{-\frac{1}{2}\left(p-\sqrt{2}(B+D)\right)^2}, \quad (34)$$

while $A, B, C, D$ in Eq. (33,34) are defined as:

$A = Re\{\tilde{g}_{\mathrm{Q2}}\}(2\alpha + 2\beta - m - n)$, $\qquad B = Im\{\tilde{g}_{\mathrm{Q2}}\}(2\alpha + 2\beta - m - n)$,

$C = Re\{\tilde{g}_{\mathrm{Q1}}\}(2\delta + 2\gamma - k - l)$, $\qquad D = Im\{\tilde{g}_{\mathrm{Q1}}\}(2\delta + 2\gamma - k - l)$.

One can use the $|\mathrm{comb}_4^0\rangle_{\mathrm{e}}$ electron as well, to create a 2D coherent lattice with fewer electrons. According to Eq. (112), after the interaction of a $|\mathrm{comb}_4^0\rangle_{\mathrm{e}}$ and a photonic mode general state, the final state is:

$$|\psi\rangle_{\mathrm{final}} = S_{g_{\mathrm{Q}}}|\mathrm{comb}_4^0\rangle_{\mathrm{e}}|\psi\rangle_{\mathrm{ph}} = \frac{1}{4}\begin{pmatrix} |\mathrm{comb}(0)\rangle_{\mathrm{e}}D_{g_{\mathrm{Q}}} + \left|\mathrm{comb}\left(\frac{\pi}{2}\right)\right\rangle_{\mathrm{e}}D_{ig_{\mathrm{Q}}} \\ +|\mathrm{comb}(\pi)\rangle_{\mathrm{e}}D_{-g_{\mathrm{Q}}} + \left|\mathrm{comb}\left(\frac{3\pi}{2}\right)\right\rangle D_{-ig_{\mathrm{Q}}} \end{pmatrix}|\psi\rangle_{\mathrm{ph}} =$$

$$= \frac{1}{4}\begin{pmatrix} |\text{comb}_4^0\rangle_{\text{e}}\left(D_{g_{\text{Q}}} + D_{ig_{\text{Q}}} + D_{-g_{\text{Q}}} + D_{-ig_{\text{Q}}}\right) \\ +|\text{comb}_4^1\rangle_{\text{e}}\left(D_{g_{\text{Q}}} + iD_{ig_{\text{Q}}} - D_{-g_{\text{Q}}} - iD_{-ig_{\text{Q}}}\right) \\ +|\text{comb}_4^2\rangle_{\text{e}}\left(D_{g_{\text{Q}}} - D_{ig_{\text{Q}}} + D_{-g_{\text{Q}}} - D_{-ig_{\text{Q}}}\right) \\ +|\text{comb}_4^3\rangle_{\text{e}}\left(D_{g_{\text{Q}}} - iD_{ig_{\text{Q}}} - D_{-g_{\text{Q}}} + iD_{-ig_{\text{Q}}}\right) \end{pmatrix}|\psi\rangle_{\text{ph}} \tag{35}$$

After post-selecting the state $|\text{comb}_4^0\rangle_{\text{e}}$ $m$ times, the final photonic state is:

$$|\psi_{\text{final}}\rangle_{\text{ph}} = |N_\psi|^{-\frac{1}{2}}\left(D_{g_{\text{Q}}} + D_{-g_{\text{Q}}} + D_{ig_{\text{Q}}} + D_{-ig_{\text{Q}}}\right)^m |\psi\rangle_{\text{ph}} =$$

$$|N_\psi|^{-\frac{1}{2}}\sum_{\alpha=0}^{m}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{m-\alpha}\binom{m}{\alpha}\binom{\alpha}{\beta}\binom{m-\alpha}{\gamma}e^{-ig_{\text{Q}}^2(2\beta-\alpha)(2\gamma+\alpha-m)}D_{g_{\text{Q}}(2\beta-\alpha)+ig_{\text{Q}}(2\gamma+\alpha-m)}|\psi_{\text{i}}\rangle_{\text{ph}} \tag{36}$$

Let us use Eq. (32) to calculate the quadrature representation in $x$ and $p$ coordinates representations of Eq. (36):

$$\langle x|\psi\rangle_{\text{ph}} \propto \sum_{\alpha=0}^{m}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{m-\alpha}\binom{m}{\alpha}\binom{\alpha}{\beta}\binom{m-\alpha}{\gamma}e^{i\left(B(A+C)+C(B+D)\right)}e^{i\sqrt{2}(B+D)\left(x-\sqrt{2}(A+C)\right)}e^{-\frac{1}{2}\left(x-\sqrt{2}(A+C)\right)^2}, \tag{37}$$

$$\langle p|\psi\rangle_{\text{ph}} \propto \sum_{\alpha=0}^{m}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{m-\alpha}\binom{m}{\alpha}\binom{\alpha}{\beta}\binom{m-\alpha}{\gamma}e^{i\left(B(A+C)+C(B+D)\right)}e^{-i\sqrt{2}(A+C)p}e^{-\frac{1}{2}\left(p-\sqrt{2}(B+D)\right)^2}, \tag{38}$$

while we defined:

$A = Re\{g_{\text{Q}}\}(2\beta-\alpha)$, $B = Im\{g_{\text{Q}}\}(2\beta-\alpha)$,
$C = Re\{ig_{\text{Q}}\}(2\gamma+\alpha-m)$, $D = Im\{ig_{\text{Q}}\}(2\gamma+\alpha-m)$.

As described in the main paper, choosing $g_Q = \sqrt{\pi/2}$ will create a magic state. In addition, post-selecting the state $|\text{comb}_4^2\rangle_{\text{e}}$ $m$ times will also create a magic state. To represent the final state in this case, we just multiply the elements in the sum in Eq. (36) by $(-1)^{m-\alpha}$.

## RM 3.3 Creating a GKP state starting from a vacuum state

As described in the main text, the ideal GKP state can be expended as a superposition of coherent states [10, 11]:

$$|\mu\rangle_{\text{ph}}^{\text{GKP}} \propto \sum_{\alpha,\beta\in\mathbb{Z}} D_{\sqrt{\frac{\pi}{2}}(2\alpha-\mu)}D_{i\sqrt{\frac{\pi}{2}}\beta}|0\rangle_{\text{ph}}, \tag{39}$$

where $\mu = 0$ and $\mu = 1$ define the logical GKP qubits $|0\rangle_{\text{ph}}^{\text{GKP}}$ and $|1\rangle_{\text{ph}}^{\text{GKP}}$ respectively.

To engineer the process of preparing an approximate rectangular GKP state, we take Eq. (30) and choose the following parameters:

$$g_{\text{Q1}} = i\sqrt{\frac{\pi}{8}}, \quad g_{\text{Q2}} = \sqrt{\frac{\pi}{2}}, \quad n = l = 0, \quad k = 4m, \tag{40}$$

meaning we induce $k=4m$ interactions with coupling constant $i\sqrt{\pi/8}$ to create a squeezed vacuum state and afterward induce another $m$ interactions with coupling constant $\sqrt{\pi/2}$ to create the approximated GKP state. In addition, we assume that all the post-selected energies are even. We substitute Eq. (40) into the general 2D grid expression from Eq. (31) and get:

$$|\mathrm{GKP}'\rangle_{\mathrm{ph}}^{m} \propto \sum_{\alpha=0}^{m}\sum_{\beta=0}^{4m}\binom{m}{\alpha}\binom{4m}{\beta} D_{\sqrt{\frac{\pi}{2}}(2\alpha-m)} D_{i\sqrt{\frac{\pi}{2}}(\beta-2m)} |0\rangle_{\mathrm{ph}}\,. \tag{41}$$

Compared to Eq. (39), we see that we get a logical GKP qubit $|0\rangle_{\mathrm{ph}}^{\mathrm{GKP}}$ or $|1\rangle_{\mathrm{ph}}^{\mathrm{GKP}}$depending on whether $m$ is even or odd, respectively. Setting $g_{Q1}, g_{Q2}, k$ according to Eq. (40) in Eq. (33, 34), we write $|\mathrm{GKP}'\rangle_{\mathrm{ph}}$ in the quadrature representation:

$$\mathrm{GKP}'(x) \propto \sum_{\alpha=0}^{m}\binom{m}{\alpha} 2^{4m}\cos^{4m}\left(\sqrt{\pi}\left(x+\sqrt{\pi}m\right)/2\right) e^{-\frac{1}{2}\left(x-\sqrt{\pi}(2\alpha-m)\right)^2}. \tag{42}$$

$$\mathrm{GKP}'(p) \propto \sum_{\beta=0}^{4m}\binom{4m}{\beta} 2^{m}\cos^{m}\left(\sqrt{\pi}p\right) e^{-\frac{1}{2}\left(p-\sqrt{\pi}(\beta-2m)\right)^2}\,. \tag{43}$$

We notice that for this special case, the peaks of the $\cos^{m}\left(\sqrt{\pi}p\right)$ term have the same frequency and phase as the Gaussian envelopes $\exp\left(-\frac{1}{2}\left(p-\sqrt{\pi}(\beta-2m)\right)^2\right.$. For this case, we get a comb-like function. An important parameter of such approximated GKP states is the squeezing parameter. We calculate the squeezing parameter of $|\mathrm{GKP}'\rangle_{\mathrm{ph}}$, by considering the probability distribution around the peak, which is the same for all peaks. For the $x$-quadrature, the probability around each peak:

$$P(x) \propto \cos^{8m}\left(\sqrt{\pi}\left(x+\sqrt{\pi}m\right)/2\right) e^{-\left(x+\sqrt{\pi}m\right)^2} \sim e^{-(1+\pi m)\left(x+\sqrt{\pi}m\right)^2} \tag{44}$$

Therefore, the variance in the $x$-quadrature and in the p- quadrature is:

$$\Delta_x^2 = \Delta_p^2 \cong \frac{1}{1+\pi m}. \tag{45}$$

To get ~10dB squeezing, defined by $S_{\mathrm{dB}} = -10\log_{10}\Delta^2$ [12], we need to choose $m=3$. Therefore, 12 interactions with $g_{\mathrm{Q1}} = i\sqrt{\pi/8}$ and 3 interactions with $g_{\mathrm{Q2}} = \sqrt{\pi/2}$ will create the desirable GKP state with 10dB squeezing.

## RM 3.4 Creating a GKP state starting from a squeezed vacuum state

Another approach for creating a GKP state is to consider the interaction of even comb electrons with a squeezed vacuum state inside the photonic mode [13]. Starting from a squeezed vacuum state will shorten the number of electrons required for 10dB squeezing, and thus increase the probability of a GKP preparation. After having $m$ interactions with even post-selections, the final state is:

$$|\psi\rangle_{\mathrm{ph}} \propto \sum_{\alpha=0}^{m}\binom{m}{\alpha} D_{g_{\mathrm{Q}}(2\alpha-m)} S(\xi)|0\rangle_{\mathrm{ph}}\,, \tag{46}$$

where $S(\xi) = e^{\frac{1}{2}\left(\xi^* \hat{a}^2 - \xi a^{\dagger 2}\right)}$ is the squeezing operator and $\xi = re^{i\theta}$ is the squeezing parameter [5]. For $g_{\mathrm{Q}} = \sqrt{\pi/2}, \theta = 0$, a rectangle GKP is prepared, and for $g_{\mathrm{Q}} = \sqrt{\pi/\sqrt{3}}, \theta = \pi/6$ a hexagonal GKP is prepared [11]. Let us calculate this state in terms of $x$ and $p$ quadratures. We use [14] to write for a general $g_{\mathrm{Q}} \coloneqq g_{\mathrm{R}} + ig_{\mathrm{I}}$ the following:

$$\langle x|D_{(g_{\mathrm{R}}+ig_I)(2\alpha-m)}|q\rangle = e^{ig_{\mathrm{R}}g_{\mathrm{I}}(2\alpha-m)^2}\langle x-\sqrt{2}g_{\mathrm{R}}(2\alpha-m)|D_{ig_{\mathrm{I}}(2\alpha-m)}|q\rangle =$$
$$= e^{ig_{\mathrm{R}}g_{\mathrm{I}}(2\alpha-m)^2} e^{i\sqrt{2}g_{\mathrm{I}}(2\alpha-m)\left(x-\sqrt{2}g_{\mathrm{R}}(2\alpha-m)\right)}\delta\left(q-\left(x-\sqrt{2}g_{\mathrm{R}}(2\alpha-m)\right)\right), \tag{47}$$

$$\langle x|S|0\rangle \propto e^{-\frac{x^2}{2}\left(\frac{1+e^{2i\theta}\tanh r}{1-e^{2i\theta}\tanh r}\right)}, \tag{48}$$

$$\langle p|S|0\rangle \propto e^{-\frac{p^2}{2}\left(\frac{1-e^{2i\theta}\tanh r}{1+e^{2i\theta}\tanh r}\right)}\sqrt{\frac{1-e^{2i\phi}\tanh r}{1+e^{2i\phi}\tanh r}}. \tag{49}$$

Thus, according to the Eq. (47-49), for a general $g_Q \coloneqq g_{\mathrm{R}} + ig_{\mathrm{I}}$, the $x$-representation is:

$$\psi(x) \propto \sum_{\alpha=0}^{m}\binom{m}{\alpha}\langle x|D_{(g_{\mathrm{R}}+ig_{\mathrm{I}})(2\alpha-m)}\int dq|q\rangle\langle q|\,S(r,\theta)|0\rangle =$$
$$= \sum_{\alpha=0}^{m}\binom{m}{\alpha} e^{ig_{\mathrm{R}}g_{\mathrm{I}}(2\alpha-m)^2} e^{i\sqrt{2}g_{\mathrm{I}}(2\alpha-m)\left(x-\sqrt{2}g_{\mathrm{R}}(2\alpha-m)\right)}\left\langle x-\sqrt{2}g_{\mathrm{R}}(2\alpha-m)\middle|S|0\right\rangle. \tag{50}$$

Using Eq. (48), one can write:

$$\psi(x) \propto \sum_{\alpha=0}^{m}\binom{m}{\alpha} e^{ig_{\mathrm{R}}g_{\mathrm{I}}(2\alpha-m)^2} e^{i\sqrt{2}g_{\mathrm{I}}(2\alpha-m)\left(x-\sqrt{2}g_{\mathrm{R}}(2\alpha-m)\right)} e^{-\frac{1}{2}\left(x-\sqrt{2}g_{\mathrm{R}}(2\alpha-m)\right)^2\left(\frac{1+e^{2i\theta}\tanh r}{1-e^{2i\theta}\tanh r}\right)}. \tag{51}$$

$p$-representation is:

$$\psi(p) \propto \sum_{\alpha=0}^{m}\binom{m}{\alpha}\langle p|D_{g_{\mathrm{Q}}(2\alpha-m)}\int d\pi|\pi\rangle\langle\pi|\,S|0\rangle =$$
$$\sum_{\alpha=0}^{m}\binom{m}{\alpha} e^{ig_{\mathrm{R}}g_{\mathrm{I}}(2\alpha-m)^2} e^{-i\sqrt{2}g_{\mathrm{R}}(2\alpha-m)p}\left\langle p-\sqrt{2}g_{\mathrm{I}}(2\alpha-m)\middle|S|0\right\rangle. \tag{52}$$

Using Eqs. (49, 52), we write:

$$\psi(p) \propto \sum_{\alpha=0}^{m}\binom{m}{\alpha} e^{ig_{\mathrm{R}}g_{\mathrm{I}}(2\alpha-m)^2} e^{-i\sqrt{2}g_{\mathrm{R}}(2\alpha-m)p}\, e^{-\frac{\left(p-\sqrt{2}g_{\mathrm{I}}(2\alpha-m)\right)^2}{2}\left(\frac{1-e^{2i\theta}\tanh r}{1+e^{2i\theta}\tanh r}\right)}. \tag{53}$$

For a real $g_Q$, one can use Eq. (51) to write:

$$\mathrm{GKP}''(x) \propto \sum_{\alpha=0}^{m}\binom{m}{\alpha} e^{-\frac{1}{2}\left(x-\sqrt{2}g_{\mathrm{Q}}(2\alpha-m)\right)^2\left(\frac{1+e^{2i\theta}\tanh r}{1-e^{2i\theta}\tanh r}\right)}, \tag{54}$$

$$\mathrm{GKP}''(p) \propto e^{-\frac{p^2}{2}\left(\frac{1-e^{2i\theta}\tanh r}{1+e^{2i\theta}\tanh r}\right)}\left(1+e^{-2ip\sqrt{2}g_{\mathrm{Q}}}\right)^m. \tag{55}$$

Similar to the case of creating GKP from a vacuum state, we will calculate the squeezing of the final state. The state probability distribution in the $x$-representation is the following: $P(x)\sim\left|\exp\left(-\frac{1}{2}\left(\frac{1+e^{2i\theta}\tanh r}{1-e^{2i\theta}\tanh r}\right)x^2\right)\right|^2$. Therefore, the variance is $\Delta_x^2 = \mathrm{Re}\left(\frac{1-e^{2i\theta}\tanh r}{1+e^{2i\theta}\tanh r}\right)$. In the $p$-direction: $P(p)\sim\left|e^{-\frac{1}{2}p^2\left(\frac{1-e^{2i\theta}\tanh r}{1+e^{2i\theta}\tanh r}\right)}e^{-mp^2 g_{\mathrm{Q}}{}^2}\right|^2$. Therefore, the variance is $\Delta_p^2 = \mathrm{Re}\left(\frac{1-e^{2i\theta}\tanh r}{1+e^{2i\theta}\tanh r}+2g_{\mathrm{Q}}{}^2 m\right)^{-1}$ while 10dB squeezing requires $\Delta^2 = 1/10$.

Let us calculate the squeezing parameters for a rectangular GKP, with $\theta = 0$ and $g_Q = \sqrt{\pi/2}$. In this case, $S_{\mathrm{dB}} = 10\log_{10}(e^{-2r}+\pi m)$. To get 10dB in the $x$-axis, $r = 1.1513$ is needed. In the $p$-axis, $m = 3.15$ is needed, while $m$ is an integer. Thus, one needs to choose $m = 4$. However, using $m = 3$ the resulting squeezing will be 9.8dB.

For the hexagonal GKP state, with $\theta = \pi/6, g_{\mathrm{Q}} = \sqrt{\pi/\sqrt{3}}$, we find the squeezing parameter in the main axes of the hexagonal grid. Taking $g_Q \rightarrow g_Q e^{it_1}$ and $\theta \rightarrow \theta + t_1$, we can rotate the GKP state by an angle $t_1$. Then we use Eqs. (54, 55) to calculate the squeezing parameters in other axes as well. For example, choosing $t_1 = -\pi/6$, the squeezing parameters will represent the projections on the $\frac{\pi}{6},\frac{2\pi}{3}$ axes. To get 10dB, we calculated the squeezing parameters in all the following axes: $0,\frac{\pi}{2},\frac{\pi}{6},\frac{2\pi}{3},\frac{\pi}{3},\frac{5\pi}{6}$. Simulating these states, one can notice that as $r$ increases, $m$ decreases, and the probability increases. We chose the squeezing parameter such that the squeezing in all X, Y, and Z directions (the directions of the Pauli operators of the hexagonal GKP) will be larger than 10dB. Four electrons are necessary for preparing 10dB squeezed GKP state. The squeezing parameter for the initial squeezed vacuum state is $r = 1.64$. This scheme has a success probability of 27.3%.

# RM 4 Probability of success for preparing the grids of coherent states

In this section, we calculate the probability to get the final photonic state according to the protocols suggested in the main text (Table 1) and described in RM 3.3 and RM 3.4.

## RM 4.1 The probability of post-selecting a specific grid of coherent states

Here, we calculate the probability of post-selecting the photonic state $|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}}$ from Eq. (28), starting from the state $|\psi_{\mathrm{i}}\rangle_{\mathrm{ph}}$, inducing $n+m$ interactions with even comb electrons and post-selectin $m$ even and $n$ odd energies.

For the 1D grid, we take an initial state of an even comb electron and a general photonic state:

$$|\Psi_{\mathrm{initial}}\rangle = |\mathrm{comb}_{\mathrm{even}}\rangle_{\mathrm{e}} \otimes |\psi_{\mathrm{i}}\rangle_{\mathrm{ph}}. \tag{56}$$

We define operators of even and odd energy post-selection: $E=\left(D_{g_\mathrm{Q}}+D_{-g_\mathrm{Q}}\right)/2$, $O=\left(D_{g_\mathrm{Q}}-D_{-g_\mathrm{Q}}\right)/2$. After one interaction with an even comb electron, the final joint state is:

$$E|\mathrm{comb_{even}}\rangle_\mathrm{e}\otimes|\psi_\mathrm{i}\rangle_\mathrm{ph}+O|\mathrm{comb_{odd}}\rangle_\mathrm{e}\otimes|\psi_\mathrm{i}\rangle_\mathrm{ph}. \tag{57}$$

According to Eq. (57), we write the final photonic state after $n+m-1$ interactions is (where $m-1$ and n represent post-selected electrons with even/odd energies respectively): $\left|\psi_{m-1,n}\right\rangle_\mathrm{ph}=\frac{O^nE^{m-1}|\psi_\mathrm{i}\rangle_\mathrm{ph}}{\|O^nE^{m-1}|\psi_\mathrm{i}\rangle_\mathrm{ph}\|}$.

The final state after another interaction with an even comb electron is:

$$\frac{1}{\left\|O^nE^{m-1}|\psi_\mathrm{i}\rangle_\mathrm{ph}\right\|}\left(O^nE^m|\mathrm{comb_{even}}\rangle_\mathrm{e}|\psi_\mathrm{i}\rangle_\mathrm{ph}+O^{n+1}E^{m-1}|\mathrm{comb_{odd}}\rangle_\mathrm{e}|\psi_\mathrm{i}\rangle_\mathrm{ph}\right). \tag{58}$$

The photonic state after post-selecting even energy is: $\left|\psi_{m,n}\right\rangle_\mathrm{ph}=\frac{O^nE^m|\psi_\mathrm{i}\rangle_\mathrm{ph}}{\|O^nE^m|\psi_\mathrm{i}\rangle_\mathrm{ph}\|}$. The probability of post-selecting this energy is: $P(\mathrm{even}|\mathrm{even}=m-1,\mathrm{odd}=n)=\left\|\frac{O^nE^m|\psi_\mathrm{i}\rangle_\mathrm{ph}}{O^nE^{m-1}|\psi_\mathrm{i}\rangle_\mathrm{ph}}\right\|^2$. Moreover, we need to sum over all the options (the different order of E and O post-selections) to post-select $\left|\psi_{m,n}\right\rangle_\mathrm{ph}$. All these options contain $m$ even post-selected energies and $n$ odd post-selected energies and prepare the same photonic state (due to the commutation relations $[E,O]=0$). There are $\binom{m+n}{m}$ options to choose $m$ even energies out of $m+n$ interactions. The probabilities for all the different routes are the same and can be calculated using a telescopic product as follows:

$$P(\mathrm{even}|\mathrm{even}=m,\mathrm{odd}=n)\cdot P(\mathrm{even}|\mathrm{even}=m-1,\mathrm{odd}=n)\cdots P(\mathrm{odd}|\mathrm{even}=0,\mathrm{odd}=0)$$
$$=\left\|\frac{O^nE^m|\psi_\mathrm{i}\rangle_\mathrm{ph}}{O^nE^{m-1}|\psi_\mathrm{i}\rangle_\mathrm{ph}}\right\|^2\left\|\frac{O^nE^{m-1}|\psi_\mathrm{i}\rangle_\mathrm{ph}}{O^nE^{m-2}|\psi_\mathrm{i}\rangle_\mathrm{ph}}\right\|^2\cdots\left\|O|\psi_\mathrm{i}\rangle_\mathrm{ph}\right\|^2=\left\|O^nE^m|\psi_\mathrm{i}\rangle_\mathrm{ph}\right\|^2. \tag{59}$$

Therefore, after substituting $O$ and $E$ operators, the final probability to have $m$ even and $n$ odd post-selected energies is:

$$P\left(|\psi_\mathrm{f}\rangle_\mathrm{ph}\right)=\binom{n+m}{m}\frac{\left\|\left(D_{g_\mathrm{Q}}-D_{-g_\mathrm{Q}}\right)^n\left(D_{g_\mathrm{Q}}+D_{-g_\mathrm{Q}}\right)^m|\psi_\mathrm{i}\rangle_\mathrm{ph}\right\|^2}{4^{n+m}}. \tag{60}$$

The coupling constant $g_\mathrm{Q}$ in this formula is a general complex number.

Using the binomial theorem, we get:

$$P\left(|\psi_\mathrm{f}\rangle_\mathrm{ph}\right)=\binom{n+m}{m}\frac{(-1)^n}{4^{n+m}}\sum_{j=0}^{2n}\sum_{i=0}^{2m}\binom{2m}{i}\binom{2n}{j}(-1)^j\left\langle\psi_\mathrm{i}\middle|D_{g_\mathrm{Q}(2m+2n-2i-2j)}\middle|\psi_\mathrm{i}\right\rangle_\mathrm{ph}. \tag{61}$$

(This equation satisfies that the sum over all probabilities of all possible states is 1.) The binomial element outside the sum counts the number of options to get the same state after post-selecting $m$ times even energies due to the commutative property of the process.

For the 2D grid, the probability of preparing the photonic state $|\psi_\mathrm{f}\rangle_\mathrm{ph}$ in Eq. (30) is:

$$P = \binom{k+l}{k}\binom{m+n}{m}\frac{\left\|\left(D_{|g_{Q2}|} - D_{-|g_{Q2}|}\right)^n \left(D_{|g_{Q2}|} + D_{-|g_{Q2}|}\right)^m \left(D_{i|g_{Q1}|} - D_{-i|g_{Q1}|}\right)^l \left(D_{i|g_{Q1}|} + D_{-i|g_{Q1}|}\right)^k |\psi_i\rangle_{ph}\right\|^2}{4^{m+n+k+l}}, \quad (62)$$

where all the other parameters $n, m, k, l$ are presented as in Eq. (30). This equation is a generalization of Eq. (60) which refers to a 1D grid. Thus, the 2D grid creation probability is obtained by multiplying the probabilities of two 1D grids. For example, we interact with the imaginary coupling constant $i|g_{Q1}|$ and obtain the probability of the photonic state. Afterward, we continue with the resulting 1D grid state and interact with a different real constant $|g_{Q2}|$, orthogonal to the 1D grid state.

For the rectangular grid of coherent states, presented in Eq. (30), the probability of success to post-select only even energies is:

$$P = \frac{1}{4^{m+k}}\left\|\left(D_{|g_{Q2}|} + D_{-|g_{Q2}|}\right)^m \left(D_{i|g_{Q1}|} + D_{-i|g_{Q1}|}\right)^k |\psi_i\rangle_{ph}\right\|^2 =$$

$$\frac{1}{4^{m+k}}\langle\psi_i|_{ph}\left(D_{i|g_{Q1}|} + D_{-i|g_{Q1}|}\right)^k \left(D_{|g_{Q2}|} + D_{-|g_{Q2}|}\right)^{2m} \left(D_{i|g_{Q1}|} + D_{-i|g_{Q1}|}\right)^k |\psi_i\rangle_{ph}. \quad (63)$$

For the case of creating a 2D grid state by $|\mathrm{comb}_4^0\rangle_e$electrons, the probability of success to post-select the state $|\mathrm{comb}_4^0\rangle_e$ $m$ times can be calculated in the same way:

$$P(|\psi_f\rangle_{ph}) = \left\|E_4^m|\psi_i\rangle_{ph}\right\|^2 = \frac{\left\|\left(D_{g_Q} + D_{-g_Q} + D_{ig_Q} + D_{-ig_Q}\right)^m|\psi_i\rangle_{ph}\right\|^2}{16^m} =$$

$$= \frac{1}{16^m}\sum_{\alpha=0}^{2m}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{2m-\alpha}\binom{2m}{\alpha}\binom{\alpha}{\beta}\binom{2m-\alpha}{\gamma} e^{-ig_Q^2(2\beta-\alpha)(2\gamma+\alpha-2m)}\langle\psi_i|_{ph}D_{g_Q(2\beta-\alpha)+ig_Q(2\gamma+\alpha-2m)}|\psi_i\rangle_{ph}. \quad (64)$$

## RM 4.2 Expanding the 1D probability formula for an initial vacuum state

The probability of post-selecting $m$ even energies and $n$ odd energies after $n+m$ is described by Eqs. (60, 61). Let us simplify Eq. (61) for the case of an initial vacuum state, i.e., $|\psi_i\rangle_{ph} = |0\rangle_{ph}$. Recalling the identity:

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(|\beta|^2+|\alpha|^2-2\beta^*\alpha)}. \quad (65)$$

Using it, one can derive:

$$P(|\psi_f\rangle_{ph}) = \binom{n+m}{m}\frac{(-1)^n}{4^{n+m}}\sum_{j=0}^{2n}\sum_{i=0}^{2m}\binom{2m}{i}\binom{2n}{j}(-1)^j e^{-2|g_Q|^2|m+n-i-j|^2}. \quad (66)$$

Since we are interested in GKP states, we will look at the special case, where only even comb electrons are post-selected ($n = 0$):

$$P\left(|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}}\right) = \frac{1}{4^m}\sum_{i=0}^{2m}\binom{2m}{i} e^{-2|g_{\mathrm{Q}}|^2|m-i|^2}. \tag{67}$$

To evaluate how fast the probability decays, we take the leading term in the sum for large $m$ and use the Stirling approximation $\binom{2m}{m} \sim \frac{2^{2m}}{\sqrt{m\pi}}$. Therefore, we get:

$$P\left(|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}}\right) \approx \frac{1}{4^m}\sum_{i=0}^{2m}\frac{2^{2m}}{\sqrt{m\pi}} e^{-2|g_{\mathrm{Q}}|^2|m-i|^2} \approx \frac{1}{\sqrt{m\pi}}\sum_{i=-\infty}^{+\infty} e^{-2|g_{\mathrm{Q}}|^2 i^2}. \tag{68}$$

Finally, we get the following expression:

$$P\left(|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}}\right) \approx \frac{1}{\sqrt{m\pi}}\theta_3\left(0, e^{-2|g_Q|^2}\right), \tag{69}$$

where $\theta_3$ is the Elliptic Theta function, which decreases for a larger $g_{\mathrm{Q}}$ and goes to 1 for $g_{\mathrm{Q}} \gg 1$. As $m$ increases, the probability decays to zero like $1/\sqrt{m\pi}$, and not exponentially as might seem at first glance from the description of the protocol (given that the first interaction is post-selected with a probability close to half). This result can be explained by the destructive interference of the wavefunction. Once the quantum light state approaches a GKP state, the operation of the O operator (defined above Eq. (57)) results in probability of odd post-selection that approaches 0.

The probability of success to post-select the state $|\mathrm{comb}_4^0\rangle_{\mathrm{e}}$ $m$ times, to create a GKP magic state, can be calculated in the same way:

$$P\left(|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}}\right) = \frac{1}{16^m}\sum_{\alpha=0}^{2m}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{2m-\alpha}\binom{2m}{\alpha}\binom{\alpha}{\beta}\binom{2m-\alpha}{\gamma} e^{-ig_{\mathrm{Q}}^2(2\beta-\alpha)(2\gamma+\alpha-2m)} e^{-\frac{1}{2}g_{\mathrm{Q}}^2\left((2\beta-\alpha)^2+(2\gamma+\alpha-2m)^2\right)} \tag{70}$$

The probability of success to measure the state $|\mathrm{comb}_4^0\rangle_{\mathrm{e}}$ $m$ times can be similarly calculated, just multiplying each element in the sum by $(-1)^{2m-\alpha}$.

## RM 4.3 The probability of post-selecting a GKP state

The probability of post-selecting the photonic state $|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}}$ for a 2D rectangular grid is described in Eq. (62). For the case of square GKP described in Eq. (41), we have:

$$P(\mathrm{GKP}') = \frac{1}{4^{5m}}\sum_{i=0}^{2m}\binom{2m}{i} e^{-\pi|m-i|^2}\sum_{j=0}^{8m}\binom{8m}{i} e^{-\frac{\pi}{4}|4m-j|^2}. \tag{71}$$

Thus, according to Eq. (68), we can use the following approximation:

$$P(\mathrm{GKP}') \approx \frac{1}{\sqrt{m\pi}}\theta_3(0, e^{-\pi})\frac{1}{\sqrt{4m\pi}}\theta_3\left(0, e^{-\frac{\pi}{4}}\right) \approx \frac{1}{m\pi}. \tag{72}$$

RM 4.4 Expanding the 1D probability formula for an initial squeezed vacuum state

In this section, we calculate the probability described by Eq. (62) for photonic mode in a squeezed vacuum = state: $|\psi_{\mathrm{i}}\rangle_{\mathrm{ph}} = S(\xi)|0\rangle_{\mathrm{ph}}$. $S(\xi)$ is the squeezing operator $e^{\frac{1}{2}\left(\xi^{*}\hat{a}^{2}-\xi a^{\dagger 2}\right)}$, where $\xi = re^{i\theta}$ is the squeezing parameter [5, 15]. We use the following property of the squeezing operator:

$$D_{\alpha}S(\xi) = S(\xi)D_{\gamma}, \qquad \gamma = \alpha\cosh r + \alpha^{*}e^{i\theta}\sinh r\,, \tag{73}$$

Thus, we get:

$$\langle\psi_{\mathrm{i}}|_{\mathrm{ph}}D_{g_{\mathrm{Q}}(2m+2n-2i-2j)}|\psi_{\mathrm{i}}\rangle_{\mathrm{ph}} = \langle 0|_{\mathrm{ph}}S^{\dagger}(\xi)D_{g_{\mathrm{Q}}(2m+2n-2i-2j)}S(\xi)|0\rangle_{\mathrm{ph}} = \langle 0|_{\mathrm{ph}}D_{\gamma}|0\rangle_{\mathrm{ph}}. \tag{74}$$

For a real coupling constant: $\gamma = g_{\mathrm{Q}}\left(\cosh r + e^{i\theta}\sinh r\right)$ and for an imaginary coupling constant: $\gamma = ig_{\mathrm{Q}}\left(\cosh r - e^{i\theta}\sinh r\right)$. Assuming $g_{\mathrm{Q}}$ is real, one can write:

$$P\left(|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}}\right) = \binom{n+m}{m}\frac{(-1)^{n}}{4^{n+m}}\sum_{j=0}^{2n}\sum_{i=0}^{2m}\binom{2m}{i}\binom{2n}{j}(-1)^{j}e^{-2\left(|g_{\mathrm{Q}}|^{2}|(m+n-i-j)|^{2}|\cosh r+e^{i\theta}\sinh r|^{2}\right)}. \tag{75}$$

Examining this equation, we notice that the relative phase between the squeezing and the coupling constant alters the probability. For the GKP case where all post-selections are even, we can further simplify and write:

$$P\left(|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}}\right) = \frac{1}{4^{m}}\sum_{i=0}^{2m}\binom{2m}{i}e^{-2|g_{\mathrm{Q}}|^{2}|(m-i)|^{2}|\cosh r+e^{i\theta}\sinh r|^{2}} \tag{76}$$

For the square GKP state ($g_{\mathrm{Q}} = \sqrt{\pi/2}\,, \theta = 0$), following Eq. (69) we derive.:

$$P\left(|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}}\right) \approx \frac{1}{\sqrt{m\pi}}\theta_{3}\left(0, e^{-\pi|\cosh r+\sinh r|^{2}}\right) \approx \frac{1}{\sqrt{m\pi}}. \tag{77}$$

# RM 5 The fidelity of the photonic state after interacting with finite electron combs

The scheme presented in our work relies on the ability to generate high-quality free-electron combs. This section analyses the effect of non-perfect combs on the resulting photonic state; it is organized in the following order: In RM 5.1, we calculate the fidelity for displacing a coherent state by a Gaussian electron comb. In RM 5.2, we calculate the fidelity for a cat state after interaction with a Gaussian electron comb. In RM 5.3, we provide a calculation for the fidelity of a 1D grid state created with $m$ consequent Gaussian electron combs, and we use that to estimate the GKP fidelity after multiple interactions. In RM 5.4, we show the dependence of the fidelity of displacing a cat on the phase of the interaction. In RM 5.5 we provide a calculation for the cat fidelity as a result of fluctuations in the coupling constant $g_{\mathrm{Q}}$ from the ideal values needed to generate the GKP state. Finally, in section RM 5.6, we consider the effect of free-space propagation and how it affects the electron comb fidelity with a perfect Gaussian electron comb.

## RM 5.1 Fidelity of the displaced coherent state generated from the interaction of a finite comb electron with a coherent state

Here we show how to calculate the fidelity of the photonic state after interacting with a finite comb electron. We assume that the initial photonic state is a coherent state, and the electron state is a finite comb with intervals $\hbar\omega$ between the peaks (imperfect $|\mathrm{comb}_1^0\rangle_\mathrm{e}$):

$$|\Psi_\mathrm{i}\rangle = D_\alpha|\psi\rangle_\mathrm{e} \otimes |0\rangle_\mathrm{ph}, \tag{78}$$

where $|\psi\rangle_\mathrm{e}$ is the finite comb-electron. The scattering matrix of the electron-photon interaction can be written in the terms of the displacement operator $S = D_{g_Q b}$. Then the final state of the electron-photon system is:

$$|\Psi_\mathrm{f}\rangle = D_{g_\mathrm{Q} b} D_\alpha|\psi\rangle_\mathrm{e} \otimes |0\rangle_\mathrm{ph}. \tag{79}$$

For an ideal comb-electron, the resulting light state is:

$$|\psi\rangle_\mathrm{ph} = D_{g_\mathrm{Q}} D_\alpha|0\rangle_\mathrm{ph} = e^{\frac{g_Q\alpha^* - g_Q^*\alpha}{2}} D_{g_\mathrm{Q}+\alpha}|0\rangle_\mathrm{ph}. \tag{80}$$

Then the expectation value for the fidelity between an imperfect comb and an ideal comb equals to:

$$F = \mathrm{Tr}_\mathrm{e}\left(\langle\psi|_\mathrm{ph}|\Psi_\mathrm{f}\rangle\langle\Psi_\mathrm{f}|\psi\rangle_\mathrm{ph}\right). \tag{81}$$

Eq. (81) is used in the following sections, where the specific post-selection changes the joint state $|\Psi_\mathrm{f}\rangle$. Eq. (81) gives the fidelity averaged over all the different options for the $|\Psi_\mathrm{f}\rangle$. Substituting Eqs. (79,80) into Eq. (81), we get:

$$\begin{aligned} F &= \langle\psi|_\mathrm{e}\langle 0|_\mathrm{ph} D_{-\alpha} D_{-g_\mathrm{Q} b} e^{\frac{g_\mathrm{Q}\alpha^* - g_\mathrm{Q}^*\alpha}{2}} D_{g_\mathrm{Q}+\alpha}|0\rangle_\mathrm{ph}\langle 0|_\mathrm{ph} e^{-\frac{g_\mathrm{Q}\alpha^* - g_\mathrm{Q}^*\alpha}{2}} D_{-g_\mathrm{Q}-\alpha} D_{g_\mathrm{Q} b} D_\alpha|0\rangle_\mathrm{ph}|\psi\rangle_\mathrm{e}. = \\ &= \langle\psi|_\mathrm{e}\langle 0|_\mathrm{ph} D_{-\alpha} D_{-g_\mathrm{Q} b} D_{g_\mathrm{Q}+\alpha}|0\rangle_\mathrm{ph}\langle 0|_\mathrm{ph} D_{-g_\mathrm{Q}-\alpha} D_{g_\mathrm{Q} b} D_\alpha|0\rangle_\mathrm{ph}|\psi\rangle_\mathrm{e} = \\ &= \langle\psi|_\mathrm{e}\langle 0|_\mathrm{ph} D_{g_\mathrm{Q}(1-b)}|0\rangle_\mathrm{ph}\langle 0|_\mathrm{ph} D_{g_\mathrm{Q}(b-1)}|0\rangle_\mathrm{ph}|\psi\rangle_\mathrm{e} = \\ &= \langle\psi|_\mathrm{e} e^{-|g_\mathrm{Q}|^2(2-b-b^\dagger)}|\psi\rangle_\mathrm{e} = e^{-2|g_\mathrm{Q}|^2}\sum_{m,n}\frac{|g_\mathrm{Q}|^{2m}|g_\mathrm{Q}|^{2n}}{m!\,n!}\langle b^{m-n}\rangle_\mathrm{e}. \end{aligned} \tag{82}$$

From this equation, we see that for the special case of coherent states, the fidelity does not depend on the coherent state amplitude $\alpha$, but only on the size of the comb and the coupling constant $g_Q$.

## RM 5.2 Fidelity of cat states following the interaction of an even finite electron comb with the vacuum

Here we show how to calculate the fidelity of a cat state after the interaction of a vacuum state with a finite even comb electron. We assume that the initial photonic mode is in a vacuum state and the electron state is a finite even comb with intervals $2\hbar\omega$ (imperfect $|\mathrm{comb}_2^0\rangle_\mathrm{e}$) between the peaks:

$$|\Psi_\mathrm{i}\rangle = |\psi\rangle_\mathrm{e} \otimes |0\rangle_\mathrm{ph}, \tag{83}$$

If we post-select even energies after the interaction, we can write the final state as:

$$|\Psi_{\rm f}\rangle \propto \left(D_{g_{\rm Q}b} + D_{-g_{\rm Q}b}\right)|\psi\rangle_{\rm e} \otimes |0\rangle_{\rm ph} = e^{-\frac{|g_{\rm Q}|^2}{2}} \sum_m \frac{g_{\rm Q}^{2m}}{\sqrt{(2m)!}} b^{2m} |\psi\rangle_{\rm e} |2m\rangle_{\rm ph}^{\rm Fock}. \tag{84}$$

For an even ideal infinite comb electron, the resulting photonic state is a cat state:

$$|{\rm cat}\rangle_{\rm ph} \propto \left|g_{\rm Q}\right\rangle_{\rm ph} + \left|-g_{\rm Q}\right\rangle_{\rm ph}. \tag{85}$$

Then the fidelity between the approximated cat state and an ideal cat state equals:

$$F = {\rm Tr}_{\rm e}\left(\langle {\rm cat}|_{\rm ph} \Psi_{\rm f}\rangle\langle\Psi_{\rm f}|{\rm cat}\rangle_{\rm ph}\right). \tag{86}$$

Substituting Eqs. (82,85) into Eq. (86), we get:

$$F \propto e^{-|g_{\rm Q}|^2} \sum_{m,n} \frac{g_{\rm Q}^{2m} {g^*_{\rm Q}}^{2n}}{\sqrt{(2m)!\,(2n)!}} \left\langle b^{2m-2n}\right\rangle_{\rm e} \left(\left\langle g_{\rm Q}\right|_{\rm ph} + \left\langle -g_{\rm Q}\right|_{\rm ph}\right) |2m\rangle_{\rm ph}^{\rm Fock} \langle 2n|_{\rm ph}^{\rm Fock} \left(\left|g_{\rm Q}\right\rangle_{\rm ph} + \left|-g_{\rm Q}\right\rangle_{\rm ph}\right) \tag{87}$$

One can further simplify this equation by using $\langle 2n|_{\rm ph}^{\rm Fock}\left(\left|g_{\rm Q}\right\rangle + \left|-g_Q\right\rangle\right) = e^{-\frac{|g_Q|^2}{2}} \frac{2g_Q^{2n}}{\sqrt{2n!}}$ and get:

$$F \propto \sum_{m,n} \frac{|g_{\rm Q}|^{4m} |g_{\rm Q}|^{4n}}{(2m)!\,(2n)!} \langle b^{2(m-n)}\rangle_{\rm e}. \tag{88}$$

To normalize the fidelity, we notice that $F = 1$ for an ideal comb, when $\langle b\rangle_{\rm e} = 1$. Thus, finally, we get:

$$F = \frac{1}{\cosh^2 |g_Q|^2} \sum_{m,n} \frac{|g_{\rm Q}|^{4m} |g_{\rm Q}|^{4n}}{(2m)!\,(2n)!} \langle b^{2m-2n}\rangle_{\rm e}\,. \tag{89}$$

We found the average fidelity of the cat state. However, in this section, we also calculate the fidelity $F_k$ after post-selecting specific electron energy $k \cdot 2\hbar\omega$, to show how the fidelity changes with the post-selected energy (Fig. 2h in the main text). For this purpose, we write the electron wavefunction on the energy basis:

$$|\psi\rangle_{\rm e} = \sum_k c_k |2k\rangle_{\rm e}\,, \tag{90}$$

where $|\psi\rangle_{\rm e}$ is the approximated electron comb with an energy difference $2\hbar\omega$, and $|2k\rangle_{\rm e}$ is the state of the electron with energy $E_0 + k \cdot 2\hbar\omega$. In this case, Eq. (84) can be simplified:

$$|\Psi_{\rm f}\rangle \propto \sum_{m,k} c_{k+m} \frac{g_Q^{2m}}{\sqrt{(2m)!}} |2m\rangle_{\rm ph} |2k\rangle_{\rm e}. \tag{91}$$

If we post-select energy $E_0 + k \cdot 2\hbar\omega$, we get the following photonic state:

$$|\psi_{\rm f}\rangle_{\rm ph} = \frac{1}{\sqrt{P(k)}} \sum_m c_{k+m} \frac{g_Q^{2m}}{\sqrt{(2m)!}} |2m\rangle_{\rm ph}^{\rm Fock}, \tag{92}$$

where $P(k) = \left|\sum_m c_{k+m} \frac{g_Q^{2m}}{\sqrt{(2m)!}}\right|^2$. In the ideal case according to Eq. (85) we have the cat state:

$$|\text{cat}\rangle_{\text{ph}} = \frac{|g_Q\rangle + |-g_Q\rangle}{\sqrt{2\left(1+e^{-2|g_Q|^2}\right)}} = \frac{1}{\sqrt{\cosh|g_{\text{Q}}|^2}} \sum_n \frac{g_Q^{2n}}{\sqrt{(2n)!}} |2m\rangle_{\text{ph}}^{\text{Fock}}. \tag{93}$$

Thus, the fidelity compared with the ideal cat state equals:

$$F(k) = |\langle\text{cat}|\psi_{\text{f}}\rangle_{\text{ph}}|^2 = \frac{1}{\cosh|g_{\text{Q}}|^2 \cdot P(k)} \left|\sum_m c_{k+m} \frac{|g_Q|^{4m}}{(2m)!}\right|^2. \tag{94}$$

The fidelity $F(k)$ for cat state is displayed in Fig. 2h in the main text. Furthermore, Eq. (94) is in full agreement with the average fidelity calculated in Eq. (89):

$$F = \frac{\sum_k F(k) \cdot P(k)}{\sum_k P(k)}, \tag{95}$$

where $F$ is the fidelity calculated according to Eq. (89) and $F(k)$ is the fidelity calculated according to Eq. (94). In all next sections, we will calculate the average fidelity only.

For calculating the fidelity we further define a Gaussian electron comb $|\psi\rangle_{\text{e}} = \frac{1}{\sqrt{\theta_3(0,\exp(-1/2\sigma_{\text{e}}^2))}} \sum_{k=-\infty}^{\infty} e^{-k^2/4\sigma_{\text{e}}^2} |2k\rangle_{\text{e}}$, such that the probability to post-select an even peak will follow the discrete Gaussian distribution $\frac{1}{\theta_3(0,\exp(-1/2\sigma_{\text{e}}^2))} e^{-k^2/2\sigma_{\text{e}}^2}$, which is normalized by the Elliptic Theta function $\theta_3$.

## RM 5.3 Fidelity estimation for GKP state prepared by a finite comb electron

The GKP state protocols require post-selection over the electron energy, each post-selected energy results in a different photonic state depending on the approximated state for an ideal comb. The probability of post-selecting different even energies for an approximated comb depends on the specific deviation from a perfect comb. First, we consider an approximated comb-electron interacting with an ideal 1D photonic grid state. If we post-select even energies after the interaction, we can write the final state as:

$$|\Psi_{\text{f}}\rangle \propto \left(D_{g_{\text{Q}}b} + D_{-g_{\text{Q}}b}\right)\left(D_{g_{\text{Q}}} + D_{-g_{\text{Q}}}\right)^m |0\rangle_{\text{ph}}|\psi\rangle_{\text{e}}, \tag{96}$$

where $\left(D_{g_{\text{Q}}} + D_{-g_{\text{Q}}}\right)^m |0\rangle_{\text{ph}}$ is an initial 1D grid state. For an ideal electron comb, the photonic state after the post-selection of even energies is:

$$|\psi\rangle_{\text{ph}} \propto \left(D_{g_{\text{Q}}} + D_{-g_{\text{Q}}}\right)^{m+1} |0\rangle_{\text{ph}}. \tag{97}$$

According to Eqs. (86, 96), the fidelity equals:

$$F \propto \langle f(b) f^\dagger(b)\rangle_{\text{e}}, \tag{98}$$

while $f(b)$ is defined as:

$$f(b) = \langle 0|_{\mathrm{ph}} \left(D_{g_{\mathrm{Q}}} + D_{-g_{\mathrm{Q}}}\right)^{m} \left(D_{g_{\mathrm{Q}}b} + D_{-g_{\mathrm{Q}}b}\right)\left(D_{g_{\mathrm{Q}}} + D_{-g_{\mathrm{Q}}}\right)^{m+1} |0\rangle_{\mathrm{ph}}, \tag{99}$$

where $\langle \dots \rangle_{\mathrm{e}} = \langle \psi | \dots | \psi \rangle_{\mathrm{e}}$ means the average over the finite electron comb. We can further simplify Eq. (99) using the binomial theorem:

$$f(b) = \langle 0|_{\mathrm{ph}} \sum_{n=0}^{m} \binom{m}{n} \left( e^{\frac{|g_{\mathrm{Q}}|^2}{2}(2n-m)(b^\dagger - b)} D_{(2n-m+b)g_{\mathrm{Q}}} + e^{-\frac{|g_{\mathrm{Q}}|^2}{2}(2n-m)(b^\dagger - b)} D_{(2n-m-b)g_{\mathrm{Q}}} \right) \left(D_{g_{\mathrm{Q}}} + D_{-g_{\mathrm{Q}}}\right)^{m+1} |0\rangle_{\mathrm{ph}} =$$

$$= \langle 0|_{\mathrm{ph}} \sum_{k,n} \binom{m+1}{k} \binom{m}{n} \left( e^{\frac{|g_{\mathrm{Q}}|^2}{2}(2n-2k+1)(b^\dagger - b)} D_{(2n+2k-2m-1+b)g_{\mathrm{Q}}} + e^{-\frac{|g_{\mathrm{Q}}|^2}{2}(2n-2k+1)(b^\dagger - b)} D_{(2n+2k-2m-1-b)g_{\mathrm{Q}}} \right) |0\rangle_{\mathrm{ph}} . \tag{100}$$

We can use the formula $\langle 0 | D_\alpha | 0 \rangle_{\mathrm{ph}} = e^{-\frac{|\alpha|^2}{2}}$, and finally, get:

$$\begin{cases} F_m = \dfrac{1}{\mathrm{norm}} \left\langle f(b) f^\dagger(b) \right\rangle_{\mathrm{e}}, \\ f(b) = \displaystyle\sum_{k,n} \binom{m+1}{k} \binom{m}{n} e^{-\frac{|g_{\mathrm{Q}}|^2 (2n+2k-2m-1)^2}{2}} \cosh\left[ |g_{\mathrm{Q}}|^2 \left( b^\dagger (2n-m) + b(2k-m-1) \right) \right], \\ \mathrm{norm} = \left| \displaystyle\sum_{k,n} \binom{m+1}{k} \binom{m}{n} e^{-\frac{|g_{\mathrm{Q}}|^2 (2n+2k-2m-1)^2}{2}} \cosh\left[ |g_{\mathrm{Q}}|^2 \left( (2n-m) + (2k-m-1) \right) \right] \right|^2 . \end{cases} \tag{101}$$

In the case of the creation of a cat state from a vacuum state ($m = 0$), Eq. (101) can be simplified:

$$F_0 = \frac{\left\langle \cosh\left( |g_{\mathrm{Q}}|^2 b \right) \cosh\left( |g_{\mathrm{Q}}|^2 b^\dagger \right) \right\rangle_{\mathrm{e}}}{\cosh^2 |g_{\mathrm{Q}}|^2} . \tag{102}$$

One can notice that this is the same result as in Eq. (89).

Now we can estimate the fidelity for GKP states after $m$ interactions with real $g_{\mathrm{Q1}}$ and $n$ interactions with imaginary $g_{\mathrm{Q2}}$. We assume that we can multiply the fidelity of each $m^{\mathrm{th}}$ step and calculate the total fidelity in both directions by multiplying the fidelities in each direction since the interactions are orthogonal. To justify this assumption, we further discuss the fidelity of displacements in orthogonal directions in section RM 5.4 and Fig. RM 1d. Under these assumptions the fidelity for a square GKP is:

$$F = \prod_{i=0}^{m} F_i^{g_{\mathrm{Q1}}} \prod_{j=0}^{n} F_j^{g_{\mathrm{Q2}}} , \tag{103}$$

where $F_i^{g_{Q1}}$ is calculated according to Eq. (101). We calculate the fidelity for a Gaussian electron comb with intervals $2\hbar\omega$ and a standard deviation of 30 peaks $\sigma_e$, for rows 2 and 3 in Table 1 of the main text (Fig RM 1a). We can see that for the required squeezing level of 10dB ($m = 3$) the fidelity is above 98%. The 3rd row of the table has a better fidelity since the number of electrons is much smaller (6 electrons for the 3rd row, compared to 15 electrons for the 2nd row).

## RM 5.4 Fidelity of a displaced cat state after the interaction with a finite comb electron

In the previous sections, we assumed that the phase of the initial state is equal to the phase of the coupling $g_{\mathrm{Q}}$. However, generally, it is not the case. Here we investigate how the fidelity depends on the phase between the initial cat state $|\alpha\rangle_{\mathrm{ph}} + |-\alpha\rangle_{\mathrm{ph}}$ and the coupling $g_Q$. For simplicity, we consider a finite comb with intervals $\hbar\omega$ (imperfect $|\mathrm{comb}_1^0\rangle_{\mathrm{e}}$) between the peaks and trace out over the electron (for an ideal comb, the electron does not entangle to the photonic state). The final state of the electron-photon system is:

$$|\Psi_{\mathrm{f}}\rangle \propto D_{g_{\mathrm{Q}}b}(D_\alpha + D_{-\alpha})|\psi\rangle_{\mathrm{e}}|0\rangle_{\mathrm{ph}}. \tag{104}$$

In the case of the ideal comb, the light state is:

$$|\psi\rangle_{\mathrm{ph}} \propto D_{g_{\mathrm{Q}}}(D_\alpha + D_{-\alpha})|0\rangle_{\mathrm{ph}}. \tag{105}$$

The fidelity is calculated according to Eq. (86). Similarly to previous sections, we can simplify the fidelity to get:

$$F = \frac{1}{\left(e^{-2|\alpha|^2}+1\right)^2}\left\langle\psi\middle| e^{|g_{\mathrm{Q}}|^2\left(b^\dagger+b-2\right)}\begin{pmatrix} e^{-2|\alpha|^2}\cosh\left(\alpha^* g_{\mathrm{Q}}(b-1)+\alpha g_{\mathrm{Q}}^*\left(b^\dagger-1\right)\right) \\ +\cosh\left(\alpha^* g_{\mathrm{Q}}(b-1)-\alpha g_{\mathrm{Q}}^*\left(b^\dagger-1\right)\right)\end{pmatrix}^2\middle|\psi\right\rangle_e . \tag{106}$$

This equation allows us to calculate the fidelity for an arbitrary $\alpha$ and an arbitrary phase between $g_Q$ and $\alpha$. The dependence of the fidelity as a function of the phase for different Gaussian electron combs is depicted in Fig. RM 1d. From Fig. RM 1d we learn that when the displacement of the electron is orthogonal to the cat state, the effect of the cat amplitude on the fidelity is minimal and a slightly higher than the fidelity of a displaced vacuum state (see section RM 5.1) This fact follows the physical intuition that the difference between parallel or orthogonal $g_{\mathrm{Q}}$ and $\alpha$, is that electron bunched in the time domain is in phase with the peaks of the oscillating electric field (when $g_{\mathrm{Q}}$ and $\alpha$ are real) or when the electric field exactly crosses zero (when $g_{\mathrm{Q}}$ is imaginary and $\alpha$ is real). We expect the field amplitude $\alpha$ will affect when it has the same phase as the electron comb. Furthermore, the higher relative fidelity for the cat state (when $g_{\mathrm{Q}}$ is imaginary and $\alpha$ is real) compared to the displaced coherent state resulting from the fact that the cat state is squeezed relative to the vacuum state.

## RM 5.5 Fidelity for creation of a cat state taking into account fluctuation in the coupling $g_Q$

Let us consider the case when we have small fluctuations in the coupling constant with a deviation $\epsilon$. According to Eq. (78) the final state after one interaction takes the following form:

$$|\Psi_{\mathrm{f}}\rangle \propto D_{(g_{\mathrm{Q}}+\epsilon)b}|\psi\rangle_{\mathrm{e}}|0\rangle_{\mathrm{ph}} \propto \sum_m \frac{\left(g_{\mathrm{Q}}+\epsilon\right)^{2m}}{\sqrt{(2m)!}} b^{2m}|\psi\rangle_{\mathrm{e}}|2m\rangle_{\mathrm{ph}}^{\mathrm{Fock}}. \tag{107}$$

For an ideal comb electron, after post-selection a cat state is generated, presented in Eq. (85). After calculating the fidelity according to Eq. (86) and repeating the same mathematical steps as in RM 5.2, we get:

$$F(\epsilon) = \frac{1}{\cosh\left(\left|g_{\mathrm{Q}}\right|^2\right)\cosh\left(\left|g_{\mathrm{Q}}+\epsilon\right|^2\right)} \sum_{m,n} \frac{\left|g_{\mathrm{Q}}\right|^{2m+2n}\left|g_{\mathrm{Q}}+\epsilon\right|^{2m+2n}}{(2m)!\,(2n)!} \langle b^{2(m-n)} \rangle_{\mathrm{e}}. \qquad (108)$$

Now let us consider the fidelity for the case where $\epsilon$ is a Gaussian random variable with a standard deviation $\Delta g_{\mathrm{Q}}$ and average 0. The probability density function is:

$$P\left(\epsilon, \Delta g_{\mathrm{Q}}\right) = \frac{1}{\Delta g_{\mathrm{Q}}\sqrt{2\pi}} e^{-\frac{1}{2\Delta g_{\mathrm{Q}}^2}\epsilon^2}. \qquad (109)$$

We can calculate the fidelity as a function of $\Delta g_{\mathrm{Q}}$ by:

$$F\left(\Delta g_{\mathrm{Q}}\right) = \int_{-\infty}^{\infty} F(\epsilon) P\left(\epsilon, \Delta g_{\mathrm{Q}}\right) d\epsilon. \qquad (110)$$

We plot the fidelity as a function of $\Delta g_{\mathrm{Q}}$ for a Gaussian comb of $\sigma_{\mathrm{e}} = 30$ and $g_{\mathrm{Q}} = \sqrt{\pi/2}$ in Fig RM 1b. We can see that the fidelity goes like $\propto 1 - \Delta g_{\mathrm{Q}}^2$, suggesting rather small changes for the cat fidelity with $\Delta g_{\mathrm{Q}}$. Such variations usually appear in grazing angle interactions since the electron beam interacts with evanescent fields that decay exponentially. In other words, the electric field $E_z(r_\perp, z-t)$ is not uniform along $r_\perp$ and t.

## RM 5.6 Free-space propagation effect on the fidelity of a Gaussian comb electron

Now, we consider the distance at which a Gaussian comb electron with $\sigma = 30$ peaks can propagate while maintaining high fidelity with itself. In order to estimate the distance, we describe the electron by the unitary evolution $U_\varphi$ for free space propagation [2]. $U_\varphi$ alters the electron state in the manner of $U_\varphi|E_k\rangle_{\mathrm{e}} = e^{-i\varphi k^2}|E_k\rangle_{\mathrm{e}}$. $\varphi$ is defined by $\varphi = 2\pi\frac{z}{z_D}$, with $z$ being the propagation distance and $z_D = 4\pi\gamma^3 m_e v^3/\hbar\omega^2$. $m_e$ is the electron mass, $v$ its velocity, and $\gamma$ is the Lorentz factor. We plot the electron fidelity of Gaussian comb electrons as a function of distance, for different average electron energies and wavelengths (Fig RM 1c)

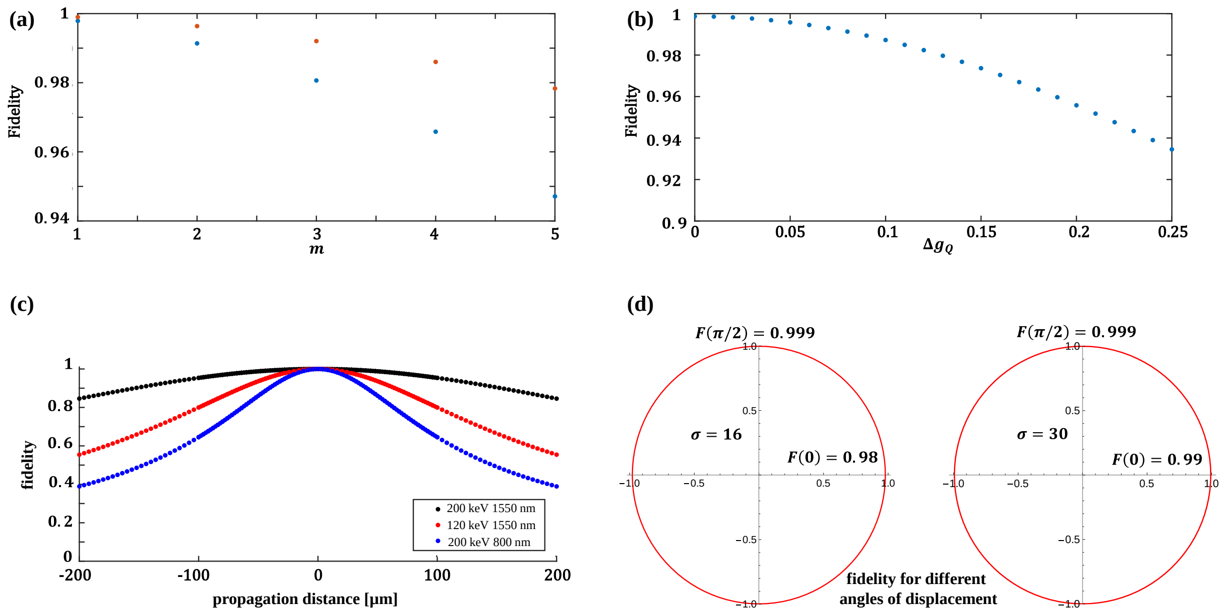


**Figure RM 1. Fidelity of GKP and cat states generated from imperfect comb electrons.** The calculations in all panels are for a Gaussian electron comb with an envelope standard deviation of $\sigma_e = 30$. **(a)** GKP fidelity as function number of steps $m$, for rows 2 (blue) and 3 (orange) in Table 1, calculated according to Eq. (103). (**b**) Cat state fidelity generated by comb electron as a function of the standard deviation of $g_Q$ (Eq. (109, 110)) normalized by $g_Q$. The coupling constant is $g_Q = \sqrt{\pi/2}$. (**c**) Fidelity of the Gaussian electron with itself after dispersion induced by free-space propagation. Electron mean energy of $E_0 = 200$ keV (black) has weaker dispersion and thus shows higher fidelity than $E_0 = 120$ keV (red), both interacting with a light of wavelength 1550 nm. Shortening the wavelength to 800 nm (blue) decreases the fidelity due to stronger dispersion. (**d**) Fidelity of a displaced cat state of $|\sqrt{\pi/2}\rangle + |-\sqrt{\pi/2}\rangle$ by $D_{e^{i\phi}\sqrt{\pi/2}}$. The fidelity is plotted in a polar plot as a function of $\phi$, demonstrating that for orthogonal displacement and cat state phases, the fidelity is not affected much by the amplitude of the cat state. However, when they are parallel to each other, the fidelity is much lower. Here the displacement fidelities compare comb electrons of $\sigma_e = 16, 30$.

### RM 5.7 Measurement noise and emitted electrons uncertainty effects on the fidelity of the final GKP state

In this section, we consider two additional error mechanisms in the creation process of the photonic state: electron detection error in the electron spectrometer and uncertainty in the number of emitted electrons, which in normal circumstances follows Poisson statistics.

As the first example, we ignore the detection error and calculate the probability to create a GKP state from a squeezed vacuum state with an unknown number of electrons per pulse, following a Poissonian distribution with parameter $\lambda$ [16, 17]:

$$P(\text{GKP from SV}) = \sum_{m \geq 3} P(\text{GKP from SV}|m \text{ electrons}) \cdot P(m \text{ electrons}). \quad (111)$$

$P(\text{GKP from SV}|m\text{ electrons})$ is given in Eq. (70) and $P(m\text{ electrons}) = \frac{\lambda^m}{m!}e^{-\lambda}$ follows the Poisson distribution with parameter $\lambda$. The probability of GKP state creation depends on the value of the $\lambda$ parameter, which we control through the intensity of the excitation laser. We find that $\lambda = 5$ optimizes the probability of GKP creation from a squeezed vacuum state, reaching $21.5\%$.

Now, we consider the electron detection efficiency and calculate the probability for an error in the GKP process taking both uncertainties into consideration. The probability of no error in the process of creating a GKP state is:

$$P(\text{no error}) = \sum_m P(\text{no error}|m\text{ electrons}) \cdot P(m\text{ electrons})\,. \tag{112}$$

$P(\text{no error}|m\text{ electrons}) = \eta^m$ is the probability to measure all the $m$ electrons correctly, with $\eta$ being the probability to correctly measure one electron. Therefore:

$$P(\text{no error}) = \sum_m \frac{(\eta \cdot \lambda)^m}{m!}e^{-\lambda} = e^{\eta\lambda}e^{-\lambda} = e^{\lambda(\eta-1)} \approx 1 - \lambda(1-\eta)\,. \tag{113}$$

Choosing $\eta = 0.99$ and $\lambda = 5$, we get $P(\text{no error}) = 0.951$.

We next evaluate the photonic density matrix with this error. The density matrix of a perfect GKP state is $\rho_{\text{no error}} = |0\rangle\langle 0|_{\text{ph}}^{\text{GKP}}$. Error in the electron detection will cause an O operator to act on the photonic mode instead of an $E$ operator (defined in Eq. (58)). Thus, the density matrix of the state with the error is:

$$\rho_{\text{error}} = E^n|0\rangle\langle 0|_{\text{ph}}E^n + OE^{n-1}|0\rangle\langle 0|_{\text{ph}}E^{n-1}O \tag{114}$$

The total density matrix is:

$$\rho = P(\text{error})\rho_{\text{error}} + P(\text{no error})\rho_{\text{no error}} \cong 0.05 \cdot \rho_{\text{error}} + 0.95 \cdot \rho_{\text{no error}} \tag{115}$$

The fidelity of this state is $0.95 \cdot F(\rho_{\text{no error}})$ with $F(\rho_{\text{no error}})$ being the fidelity of the GKP state, which considers the imperfect electron wavefunction (Eq. (103)). Thus, we found a lower bound for the fidelity of the final GKP state, taking into consideration all the possible noise channels – uncertainty in the electron number, imperfect electron preparation, and electron detection errors.

# RM 6 Creating entangled GKP states and cat states

## RM 6.1 Creating entangled cat states

As described in the main paper, we use the approach from ref. [18] to create a Bell state between two photonic modes. The initial state of the full system contains the even comb electron and two photonic modes in a vacuum state:

$$|\Psi_{\text{i}}\rangle = |\text{comb}_{\text{even}}\rangle_{\text{e}}|0\rangle_{\text{ph1}}|0\rangle_{\text{ph2}}. \tag{116}$$

After each interaction of the electron with the two photonic modes, the final state is evaluated using the scattering matrices (SM1):

$$|\Psi_{\text{f}}\rangle = S_2 S_1|\Psi_{\text{i}}\rangle. \tag{117}$$

Using the formalism presented in Eq. (57), the state after the first interaction is:

$$|\Psi_{\mathrm{f}}\rangle = S_2\big(E_1|\mathrm{comb}_{\mathrm{even}}\rangle_{\mathrm{e}}|0\rangle_{\mathrm{ph1}}|0\rangle_{\mathrm{ph2}} + O_1|\mathrm{comb}_{\mathrm{odd}}\rangle_{\mathrm{e}}|0\rangle_{\mathrm{ph1}}|0\rangle_{\mathrm{ph2}}\big), \tag{118}$$

While $E_1, O_1$ act on the first state of light, creating even and odd cat states, respectively:

$$E|0\rangle_{\mathrm{ph}} = |\mathrm{cat}_{\mathrm{even}}\rangle_{\mathrm{ph}}, \qquad O|0\rangle_{\mathrm{ph}} = |\mathrm{cat}_{\mathrm{odd}}\rangle_{\mathrm{ph}}. \tag{119}$$

Eq. (118) can be further developed, using the commutation relations $[S_2, E_1] = [S_2, O_1] = 0$ to write:

$$\begin{aligned} |\Psi_{\mathrm{f}}\rangle &= \big(E_1 S_2|\mathrm{comb}_{\mathrm{even}}\rangle_{\mathrm{e}}|0\rangle_{\mathrm{ph1}}|0\rangle_{\mathrm{ph2}} + O_1 S_2|\mathrm{comb}_{\mathrm{odd}}\rangle_{\mathrm{e}}|0\rangle_{\mathrm{ph1}}|0\rangle_{\mathrm{ph2}}\big) = \\ &= (E_2E_1 + O_2O_1)|\mathrm{comb}_{\mathrm{even}}\rangle_{\mathrm{e}}|0\rangle_{\mathrm{ph1}}|0\rangle_{\mathrm{ph2}} + (O_2E_1 + E_2O_1)|\mathrm{comb}_{\mathrm{odd}}\rangle|0\rangle_{\mathrm{ph1}}|0\rangle_{\mathrm{ph2}}. \end{aligned} \tag{120}$$

Using Eq. (119), the final joint state is:

$$\begin{aligned} |\Psi_{\mathrm{f}}\rangle = |\mathrm{comb}_{\mathrm{even}}\rangle_{\mathrm{e}}\big(|\mathrm{cat}_{\mathrm{even}}, \mathrm{cat}_{\mathrm{even}}\rangle_{\mathrm{ph}} + |\mathrm{cat}_{\mathrm{odd}}, \mathrm{cat}_{\mathrm{odd}}\rangle_{\mathrm{ph}}\big) + \\ + |\mathrm{comb}_{\mathrm{odd}}\rangle_{\mathrm{e}}\big(|\mathrm{cat}_{\mathrm{odd}}, \mathrm{cat}_{\mathrm{even}}\rangle_{\mathrm{ph}} + |\mathrm{cat}_{\mathrm{even}}, \mathrm{cat}_{\mathrm{odd}}\rangle_{\mathrm{ph}}\big). \end{aligned} \tag{121}$$

Post-selecting the electron will create a Bell state of two cat states.

## RM 6.2 Creating entangled GKP states

Using the same approach as in RM 6.1, we can entangle two GKP states into a GKP Bell state. We consider two GKP states and the 4-component comb-electron state $|\mathrm{comb}_4^0\rangle_{\mathrm{e}}$, as defined in Eq. (10). After the interaction, the final joint state is:

$$|\psi_{\mathrm{final}}\rangle = S_2 S_1|\mathrm{comb}_4^0\rangle_{\mathrm{e}}|0\rangle_{\mathrm{ph1}}^{\mathrm{GKP}}|0\rangle_{\mathrm{ph2}}^{\mathrm{GKP}}. \tag{122}$$

$|\mathrm{comb}_4^0\rangle_{\mathrm{e}}$ can be written as a superposition of 4 comb electrons with different phases:

$$\begin{aligned} |\mathrm{comb}_4^0\rangle_{\mathrm{e}} &= \frac{1}{4}(|\mathrm{comb}(0)\rangle_{\mathrm{e}} + |\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} + |\mathrm{comb}(\pi)\rangle_{\mathrm{e}} + |\mathrm{comb}(3\pi/2)\rangle_{\mathrm{e}}), \\ |\mathrm{comb}_4^1\rangle_{\mathrm{e}} &= \frac{1}{4}(|\mathrm{comb}(0)\rangle_{\mathrm{e}} - i|\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} - |\mathrm{comb}(\pi)\rangle_{\mathrm{e}} + i|\mathrm{comb}(3\pi/2)\rangle_{\mathrm{e}}), \\ |\mathrm{comb}_4^2\rangle_{\mathrm{e}} &= \frac{1}{4}(|\mathrm{comb}(0)\rangle_{\mathrm{e}} - |\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} + |\mathrm{comb}(\pi)\rangle_{\mathrm{e}} - |\mathrm{comb}(3\pi/2)\rangle_{\mathrm{e}}), \\ |\mathrm{comb}_4^3\rangle_{\mathrm{e}} &= \frac{1}{4}(|\mathrm{comb}(0)\rangle_{\mathrm{e}} + i|\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} - |\mathrm{comb}(\pi)\rangle_{\mathrm{e}} - i|\mathrm{comb}(3\pi/2)\rangle_{\mathrm{e}}). \end{aligned} \tag{123}$$

For $g_{\mathrm{Q}} = \sqrt{\pi/2}$, we can use the GKP's symmetry and Eq. (32) to achieve:

$$\begin{aligned} D_{\pm\sqrt{\pi/2}}|0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} &= e^{\mp i\sqrt{\pi}\hat{p}}|0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} = X|0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} = |1\rangle_{\mathrm{ph}}^{\mathrm{GKP}}, \\ D_{\pm i\sqrt{\pi/2}}|0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} &= e^{\pm i\sqrt{\pi}\hat{x}}|0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} = Z|0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} = |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} \end{aligned} \tag{124}$$

Thus, according to Eqs. (112, 113), after one interaction of a $|\mathrm{comb}_4^0\rangle_{\mathrm{e}}$ and a $|0\rangle_{\mathrm{ph}}^{\mathrm{GKP}}$ state with a coupling constant $g_{\mathrm{Q}} = \sqrt{\pi/2}$, the resulting joint state is:

$$S|\mathrm{comb}_4^0\rangle_{\mathrm{e}}|0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} =$$

$$= \frac{1}{4}\begin{pmatrix} |\mathrm{comb}(0)\rangle_{\mathrm{e}} D_{\sqrt{\pi/2}} + |\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} D_{i\sqrt{\pi/2}} \\ +|\mathrm{comb}(\pi)\rangle_{\mathrm{e}} D_{-\sqrt{\pi/2}} + |\mathrm{comb}(3\pi/2)\rangle D_{-i\sqrt{\pi/2}} \end{pmatrix} |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} =$$

$$= \frac{1}{4}\begin{pmatrix} |\mathrm{comb}(0)\rangle_{\mathrm{e}} X + |\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} \mathrm{Z} \\ +|\mathrm{comb}(\pi)\rangle_{\mathrm{e}} \mathrm{X} + |\mathrm{comb}(3\pi/2)\rangle \mathrm{Z} \end{pmatrix} |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} =$$

$$= \frac{1}{4}\Big( (|\mathrm{comb}(0)\rangle_{\mathrm{e}} + |\mathrm{comb}(\pi)\rangle_{\mathrm{e}}) |1\rangle_{\mathrm{ph}}^{\mathrm{GKP}} + (|\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} + |\mathrm{comb}(3\pi/2)\rangle) |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} \Big) =$$

$$= \frac{1}{4}\Big( (|\mathrm{comb}_4^0\rangle_{\mathrm{e}} + |\mathrm{comb}_4^2\rangle_{\mathrm{e}}) |1\rangle_{\mathrm{ph}}^{\mathrm{GKP}} + (|\mathrm{comb}_4^0\rangle_{\mathrm{e}} - |\mathrm{comb}_4^2\rangle_{\mathrm{e}}) |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} \Big) =$$

$$\frac{1}{2\sqrt{2}}\big( |\mathrm{comb}_4^0\rangle_{\mathrm{e}} |+\rangle_{\mathrm{ph}}^{\mathrm{GKP}} - |\mathrm{comb}_4^2\rangle_{\mathrm{e}} |-\rangle_{\mathrm{ph}}^{\mathrm{GKP}} \big). \tag{125}$$

We can interpret this result as a conditional rotation on the GKP state, while the photonic state is conditioned by the electron post-selection result. Similarly, using Eqs. (123, 124) we derive for a comb electron $|\mathrm{comb}_4^2\rangle$:

$$S|\mathrm{comb}_4^2\rangle_{\mathrm{e}} |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} =$$

$$= \frac{1}{4}\begin{pmatrix} |\mathrm{comb}(0)\rangle_{\mathrm{e}} D_{\sqrt{\pi/2}} - |\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} D_{i\sqrt{\pi/2}} \\ +|\mathrm{comb}(\pi)\rangle_{\mathrm{e}} D_{-\sqrt{\pi/2}} - |\mathrm{comb}(3\pi/2)\rangle D_{-i\sqrt{\pi/2}} \end{pmatrix} |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} =$$

$$= \frac{1}{4}\begin{pmatrix} |\mathrm{comb}(0)\rangle_{\mathrm{e}} X - |\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} \mathrm{Z} \\ +|\mathrm{comb}(\pi)\rangle_{\mathrm{e}} \mathrm{X} - |\mathrm{comb}(3\pi/2)\rangle \mathrm{Z} \end{pmatrix} |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} =$$

$$= \frac{1}{4}\Big( (|\mathrm{comb}(0)\rangle_{\mathrm{e}} + |\mathrm{comb}(\pi)\rangle_{\mathrm{e}}) |1\rangle_{\mathrm{ph}}^{\mathrm{GKP}} - (|\mathrm{comb}(\pi/2)\rangle_{\mathrm{e}} + |\mathrm{comb}(3\pi/2)\rangle) |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} \Big) =$$

$$= \frac{1}{4}\Big( (|\mathrm{comb}_4^0\rangle_{\mathrm{e}} + |\mathrm{comb}_4^2\rangle_{\mathrm{e}}) |1\rangle_{\mathrm{ph}}^{\mathrm{GKP}} - (|\mathrm{comb}_4^0\rangle_{\mathrm{e}} - |\mathrm{comb}_4^2\rangle_{\mathrm{e}}) |0\rangle_{\mathrm{ph}}^{\mathrm{GKP}} \Big) =$$

$$\frac{1}{2\sqrt{2}}\big( |\mathrm{comb}_4^2\rangle_{\mathrm{e}} |+\rangle_{\mathrm{ph}}^{\mathrm{GKP}} - |\mathrm{comb}_4^0\rangle_{\mathrm{e}} |-\rangle_{\mathrm{ph}}^{\mathrm{GKP}} \big). \tag{126}$$

From Eq. (126), one can notice that the probability for odd post-selections ($|\mathrm{comb}_4^1\rangle$, $|\mathrm{comb}_4^3\rangle$) is 0. Thus, according to Eqs. (125, 126), after the interaction of the same comb electron $|\mathrm{comb}_4^0\rangle_{\mathrm{e}}$ with two different photonic modes, the final state is:

$$|\Psi_{\mathrm{f}}\rangle = S_2 S_1 |\mathrm{comb}_4^0\rangle_{\mathrm{e}} |0\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |0\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} =$$

$$= \frac{1}{2\sqrt{2}} S_2 \big( |\mathrm{comb}_4^0\rangle_{\mathrm{e}} |+\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |0\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} - |\mathrm{comb}_4^2\rangle_{\mathrm{e}} |-\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |0\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} \big) =$$

$$= \frac{1}{8}\begin{pmatrix} \big( |\mathrm{comb}_4^0\rangle_{\mathrm{e}} |+\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |+\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} + |\mathrm{comb}_4^2\rangle_{\mathrm{e}} |+\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |-\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} \big) \\ -\big( -|\mathrm{comb}_4^0\rangle_{\mathrm{e}} |-\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |-\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} + |\mathrm{comb}_4^2\rangle_{\mathrm{e}} |-\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |+\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} \big) \end{pmatrix} =$$

$$= \frac{1}{8}\begin{pmatrix} |\mathrm{comb}_4^0\rangle_{\mathrm{e}} \big( |+\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |+\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} + |-\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |-\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} \big) \\ +|\mathrm{comb}_4^2\rangle_{\mathrm{e}} \big( |+\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |-\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} - |-\rangle_{\mathrm{ph1}}^{\mathrm{GKP}} |+\rangle_{\mathrm{ph2}}^{\mathrm{GKP}} \big) \end{pmatrix} \tag{127}$$

Finally, upon post-selection of the electron energy, the result is a GKP Bell state between the two photonic modes, as described in the main text, Eq. (15).

## RM 7 Quantum gates for the GKP state with electron combs

Our method allows not only to create GKP states but also to perform different quantum gates on them, including stabilizers, i.e., gates that do not change the ideal GKP states and help in quantum error-correction schemes. For an ideal GKP state, the stabilizers are:

$$s_1 = D_{\pm\sqrt{2\pi}}, \qquad s_2 = D_{\pm\sqrt{2\pi}i}. \tag{128}$$

$X$-gate can be found by the following formula:

$$X = D_{\pm\sqrt{\pi/2}}. \tag{129}$$

$Z$-gate can be found by the following formula:

$$Z = D_{\pm i\sqrt{\pi/2}}. \tag{130}$$

All these gates and stabilizers can be created using interactions with comb electrons with the coupling $g_{\mathrm{Q}} = \pm\sqrt{2\pi}, \pm\sqrt{2\pi}i$ for the stabilizers, $g_{\mathrm{Q}} = \pm\sqrt{\pi/2}$ for $X$ gate, and $g_{\mathrm{Q}} = \pm i\sqrt{\pi/2}$ for $Z$ gate. However, such operations significantly decrease the fidelity of the created (non-ideal) GKP state. To create the operation, which will increase the fidelity of the GKP state, we use post-selection. We need to interact with an even comb-electron $|\mathrm{comb_{even}}\rangle$ with a coupling parameter $g_{\mathrm{Q}}$ and post-select even energies. As described in this document, the light state due to this interaction will change in the following way:

$$|\psi_{\mathrm{f}}\rangle_{\mathrm{ph}} = \left(D_{g_{\mathrm{Q}}} + D_{-g_{\mathrm{Q}}}\right)|\psi_{\mathrm{i}}\rangle_{\mathrm{ph}}. \tag{131}$$

For $g_{\mathrm{Q}} = \sqrt{2\pi}$ and $g_{\mathrm{Q}} = i\sqrt{2\pi}$, we get the stabilizers of GKP states. For $g_{\mathrm{Q}} = \sqrt{\pi/2}$, we get the $X$ gate, and for $g_{\mathrm{Q}} = i\sqrt{\pi/2}$, we get the $Z$ gate. If we want to implement a $Y$ gate, we can apply $X$ and $Z$ gates consequently (i.e., $X \cdot Z$, which gives $Y$ gate up to a global phase), or use $g_{\mathrm{Q}} = \sqrt{\pi/2} + i\sqrt{\pi/2}$ with an even post-selection.

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