Comment on "Nonlinear quantum effects in electromagnetic radiation of a vortex electron"

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This comment on the paper by Karlovets and Pupasov-Maksimov [Phys. Rev. A **103**, 012214 (2021)] addresses their criticism of the combined experimental and theoretical study by Remez *et al.* [Phys. Rev. Lett. **123**, 060401 (2019)]. We show, by means of simple optical arguments as well as numerical simulations, that the arguments raised by Karlovets and Pupasov-Maksimov do not hold in the experimental regime reported by Remez *et al.* Further, we discuss a clarification for the theoretical derivations presented by Karlovets and Pupasov-Maksimov, as they hold only when the final state of the emitting electron is observed in coincidence with the emitted photon. Although this scenario is feasible and may stimulate new experimental regimes that do correspond to the predictions reported by Karlovets and Pupasov-Maksimov, it is not the common scenario in cathodoluminescence, where only the light is measured. Upon lifting the concerns regarding the experimental regime reported by Remez *et al.*, and explicitly clarifying the electron postselection, we believe that the paper by Karlovets and Pupasov-Maksimov may constitute a valuable contribution to the problem of spontaneous emission by shaped electron wave functions, as it presents new expressions for the emission rates beyond the ubiquitous paraxial approximation.

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I. INTRODUCTION

The new derivations presented in Ref. [1] are novel and interesting, and address quantum-mechanical corrections to radiation by free electrons beyond the paraxial approximation for electron beams, which is often employed in the literature. This research follows the path of previous papers on the topic of spontaneous emission by free-electron wave functions with orbital angular momentum [2,3].

However, in the start of their paper, the authors express doubts regarding the experimental results and conclusions in a combined experimental and theoretical work published by Remez *et al.* [4], where no dependence on the transverse part of the electron wave function was reported upon spontaneous emission of Smith-Purcell radiation from a uniform grating. Further, we believe the authors of Ref. [1] should clarify that their calculations hold for the physical scenario of postselection (or coincident detection) of the electron and the light.

The main point of controversy is the interpretation of the free-electron wave function in the Smith-Purcell effect [4]. In particular, Ref. [1] argues against the validity of the experimental parameters and conclusions reported in Ref. [4]. We identify the two assumptions in Ref. [1] that lead to (i) criticism regarding the validity of the far-field measurements in Ref. [4] and (ii) different treatment of quantum decoherence of the Smith-Purcell effect in the experiment, and any free-electron radiation process. These assumptions, which we believe do not hold for the experiment in Ref. [4], explain the conflicts between Refs. [1,4]. Below, we explain what in our opinion are the necessary physical assumptions in two ways: using simple optical considerations, and numerical simulations, both clarifying why the conclusions in Ref. [4] hold.

II. RECAP OF REMEZ *et al.* [4]

Before we delve into the quantitative estimate, we make a short reminder about the context of the study reported by Remez *et al.* [4]. The experiment and complementing theory aimed to decide between two possible interpretations of the role of a free electron's wave function in the process of spontaneous emission from that electron: (1) *the semiclassical interpretation*: the electron emits light coherently as a smeared-out charge density given by $e|\psi(\mathbf{r})|^2$; (2) *the quantum interpretation*: the electron effectively "collapses" to a point $\mathbf{r}$ upon emission with probability $|\psi(\mathbf{r})|^2$, and then emits light incoherently from different points (or, localized regions of size much smaller than the emitted wavelength). In Ref. [4], it was found that both experiment and QED-based theory agree with the second interpretation.

III. THEORY

(1) Semiclassical:

In the semiclassical theory, the electron current density is written as

$$\mathbf{j}(\mathbf{r},t) = e\mathbf{v}_0\delta(z - v_0t)|\psi_T(\mathbf{r}_T)|^2. \tag{1}$$

According to Jackson [5] the emitted power spectrum measured by an observer in a direction $\hat{\mathbf{n}}$ is

$$\frac{d^2P}{d\Omega d\omega} = \left|\int d^2\mathbf{r}_T|\psi_T(\mathbf{r}_T)|^2 e^{-i\frac{\omega}{c}\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}_T}\right|^2 \frac{d^2P^{\text{class}}}{d\Omega d\omega}. \tag{2}$$

If the electron state is an incoherent mixture of different wave functions, each coherent only with itself, an incoherent summation over a classical probability $p(\mathbf{r})$ is performed (see derivation in Appendix A):

$$\frac{d^2P}{d\Omega d\omega} = \int d^2\mathbf{r}_0 p(\mathbf{r}_0)\left|\int d^2\mathbf{r}_T|\psi_T(\mathbf{r}_T-\mathbf{r}_0)|^2 e^{-i\frac{\omega}{c}\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}_T}\right|^2 \times \frac{d^2P^{\text{class}}}{d\Omega d\omega}. \quad (3)$$

In both cases, the radiation divergence will be dominated by the width of $|\psi_T(\mathbf{r}_T-\mathbf{r}_0)|^2$, as the integral $\int d^2\mathbf{r}_T|\psi_T(\mathbf{r}_T)|^2 e^{-i\frac{\omega}{c}\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}_T}$ results in a narrow distribution for the direction $\hat{\mathbf{n}}$.

(2) Quantum:

Using a quantum electrodynamical calculation, and when the electron's final state is not measured, the power spectrum becomes

$$\frac{d^2P}{d\Omega d\omega} = \int d^2\mathbf{r}_T|\psi_T(\mathbf{r}_T)|^2\frac{d^2P^{\text{class}}}{d\Omega d\omega}(\mathbf{r}_T), \quad (4)$$

if the electron is in a pure state with a wave function $\psi(\mathbf{r})$, or, more generally (see derivation in Appendix B),

$$\frac{d^2P}{d\Omega d\omega} = \int d^2\mathbf{r}_T\rho_{\text{el},T}(\mathbf{r}_T,\mathbf{r}_T)\frac{d^2P^{\text{class}}}{d\Omega d\omega}(\mathbf{r}_T), \quad (5)$$

if the electron has a density matrix $\rho_{\text{el}}(\mathbf{r},\mathbf{r}')$ (i.e., it is in a mixed state). Note that the two results coincide when $\rho_{\text{el}}(\mathbf{r},\mathbf{r}')=\psi(\mathbf{r})\psi^*(\mathbf{r}')$ is a pure state.

IV. EXPERIMENT

To test the hypotheses, the azimuthal radiation pattern of Smith-Purcell radiation, emitted by narrow and wide electron beams, was measured. If the semiclassical theory was true, coherent interference from the charge density $e|\psi(\mathbf{r})|^2$ would have resulted in collimated light in the far field, with a divergence angle of

$$\theta \propto \frac{\lambda}{d}, \quad (6)$$

where d is the transverse aperture from which the light is emitted coherently, corresponding to the width of the single-electron wave function $\psi(\mathbf{r})$ [see Eqs. (2) and (3) and further discussion herein]. By changing the width of the beam via magnification, the transverse coherence of the wave function is also changed [6], and the divergence of the light should have changed accordingly.

On the other hand, if the quantum theory was true, then the light would always be azimuthally divergent, as it was emitted by a point charge [see Eqs. (4) and (5)]. Therefore, no change in azimuthal divergence should be measured when the width of the wave function is changed.

Note that, since electron beams can be partially coherent (i.e., comprising a mixture of single-electron wave functions that are coherent with themselves up to a transverse coherence length l_c), the incoherent width of the entire beam does not represent the aperture d that is considered above. Instead, the aperture of light emission from a single electron must be

$$d = l_c, \quad (7)$$

and, therefore,

$$\theta \propto \frac{\lambda}{l_c}, \quad \text{if semiclassical.} \quad (8)$$

This observation is asserted by the semiclassical numerical finite-difference time domain (FDTD) simulations in Fig. 1.

V. CONDITIONS FOR TESTING THE HYPOTHESES

The hypotheses could be tested only if the following conditions are met:

(1) *Enough spatial coherence*. The transverse coherence length l_c should be larger than the wavelength (to observe collimation, if there is one). In the experiment in Ref. [4], the measured coherence length was indeed

$$l_c = \begin{cases} 5\ \mu\text{m}, & \text{for beam width of } 300\ \mu m \\ 33\ \mu\text{m}, & \text{for beam width of } 2000\ \mu m \end{cases} \gg \lambda = 0.6\ \mu\text{m}.$$

(2) *Far-field detection*. The distance to the detector z_{det} exceeds the Fraunhofer distance (to perform the measurement in the far field). The Fraunhofer distance z_{F} corresponding to the transverse coherence length of the beam was

$$z_{\text{F}} = \frac{d^2}{\lambda} = \frac{l_c^2}{\lambda} = \begin{cases} 42\ \mu\text{m}, & 300\ \mu\text{m beam} \\ 1.8\ \text{mm}, & 2000\ \mu\text{m beam} \end{cases} \ll z_{\text{det}} = 25\ \text{cm},$$

and similarly for the coherent interaction length of the electron with the grating, reported in an earlier experiment [7] on the same setup and same conditions to be $L_{\text{int}} = 10\ \mu$m:

$$z_{\text{F}} = \frac{d^2}{\lambda} = \frac{L_{\text{int}}^2}{\lambda} = 167\ \mu\text{m} \ll z_{\text{det}} = 25\ \text{cm}.$$

See also further discussion on this point in the following section.

(3) *Low current*. The current needs to be low enough to avoid coherent effects from multiple electrons. The experiment of Ref. [4] uses a dc electron beam wherein electrons are randomly positioned in the beam and the light emission needs to be ensemble averaged over the random positions. For the reported current of $I = 40.8$ nA, less than one electron interacts with the sample along the coherent interaction length $L_{\text{int}} = 10\ \mu$m, and approximately five electrons are interacting with the sample along the incoherent interaction length of $\sim$4 mm. However, these electrons are sparsely and randomly positioned along the beam, and the random distance between them much exceeds the emitted wavelength. Therefore, coherent interference effects are expected to wash out. We assert this observation using numerical simulations of the ensemble-averaged beam structure factor in Appendix C and presented in Fig. 3.

Let us now find the critical current for which many-body effects become dominant, and show that it cannot be realized with dc currents as in the experiment of Ref. [4]. The critical current above which many-electron coherent effects become dominant, and the radiation is dictated by the whole beam size, is (see Appendix C)

$$I_0 = q^2 Wev, \quad (9)$$

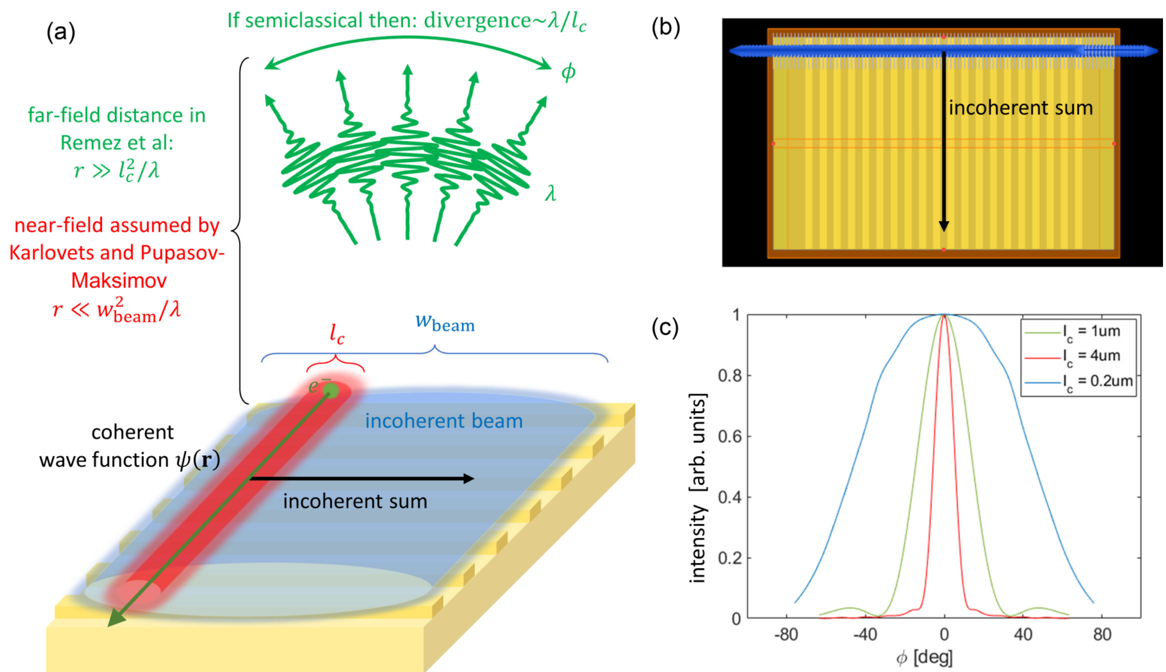

FIG. 1. Numerical FDTD simulations of semiclassical Smith-Purcell radiation by partially coherent electron beams. (a) Illustration of the Smith-Purcell emission from a partially coherent source—under the semiclassical (charge density) interpretation. The incoherent electron beam width w_{beam} is much larger than the wave function coherence length l_c. Light with wavelength λ is emitted coherently from each aperture l_c, and the radiation is summed incoherently over the beam width. A far-field observer located at $r \gg l_c^2/\lambda$ measures the light divergence along the azimuthal angle ϕ. If the light emission is semiclassical (i.e., light is emitted coherently from a smeared-out charge density), the observer should see a divergence that scales with λ/l_c. In the experiment reported in Ref. [4] this logic was used to rule out the semiclassical interpretation in favor of a quantum probabilistic point-charge model, as no change in the divergence was observed for a corresponding change in l_c. (b) FDTD model used for the semiclassical simulations. For each simulation, a coherence length was chosen and the emitted light was summed incoherently from different parts of the beam width. (c) The angular spectrum along the azimuthal angle as predicted by the semiclassical interpretation for different coherence lengths [0.2 μm: outermost (blue) line; 1 μm: middle (green) line; and 4 μm: innermost (red) line]. The semiclassical theory predicts a considerable change in the angular divergence when l_c is varied. This result was obtained upon propagating the near field to a distance of $r = 100$ μm away from the source, satisfying $r \gg l_c^2/\lambda$ but at the same time ensuring $r \ll w_{\text{beam}}^2/\lambda$—a similar regime to that reported in the experiment [4]. In contrast, the authors of Ref. [1] considered the Fraunhofer condition to be $r \gg w_{\text{beam}}^2/\lambda$, suggesting that the measurements in Ref. [4] were taken in the near field. They further claim that in this case, the radiation angular divergence should be wide for all values of l_c due to a near-field effect. Our simulations show that this is not fulfilled, and that the radiation patterns strongly depend on l_c even in the regime which the authors of Ref. [1] considered as the near field.

where q is the emitted light wave number and W the full (incoherent) beam width. For the parameters of the experiment in Ref. [4], the critical current I_0 is in excess of 1 A, which of course is orders of magnitude larger than the reported current of $I = 40.8$ nA.

While a critical value of 1 A seems rather large, note, however, that this situation is completely plausible for experiments with electron *bunches* rather than dc currents. In such cases, the peak current is $I = Q/\Delta t$ where Q is the bunch charge and Δt its duration. For example, a bunched-beam experiment [8] reported a bunch charge $Q = 4$ pC and bunch length $\Delta t = 250$ fs, giving a peak current of $I = 16$ A. In this regime, it is well known [9] that the coherent emission of radiation is dictated by the structure of the bunch rather than by the single-electron wave packet.

(4) *Paraxiality*. To avoid corrections due to the beam being nonparaxial, the variance of the transverse momentum should be much smaller than the carrier momentum of the wave packet along the azimuthal direction. In other words, the convergence angle should be small. The experiment in Ref. [4] was performed in parallel illumination of the transmission electron microscope (TEM), imaged onto the viewing chamber plane with a convergence angle of less than 0.36 μrad.

VI. VALIDITY OF THE MEASUREMENTS AND CONCLUSIONS IN REMEZ *et al.* [4]

A. Semiclassical and quantum theories of spontaneous emission by free electrons

Reference [1] claims that a semiclassical interpretation could still explain the physics of the experiment reported in Ref. [4]—and that its results cannot reject the semiclassical interpretation. The main reason cited by the authors of Ref. [1] is that the experiment in Ref. [4] was performed in near-field conditions. It is the purpose of the following section to show that the experiment reported in Ref. [4] *rejects the semiclassical theory* and that *it was performed in the far field as claimed in* Ref. [4]. We explicitly show that the semiclassical and quantum interpretations predict dramatically different radiation patterns. To do this, below we first consider the predictions made by the semiclassical interpretation—removing the ambiguity suggested by Ref. [1]—and then explain how, based on the measurements reported in Ref. [4], the semiclassical interpretation can be rejected in favor of the quantum interpretation.

B. Fraunhofer diffraction condition of light emitted by partially coherent electron beams

The authors of Ref. [1] claim that the experimental results reported in Ref. [4] do not agree with their theoretical prediction because the radiation pattern was presumably not measured in the far field. Based on this argument, they claim that the divergence observed in the radiation can still be accounted for via a semiclassical near-field effect. Put in other words, the authors argue that the Fraunhofer diffraction condition $r \gg d^2/\lambda$ [Eq. (6) in their paper] is not met in the measurements reported in Ref. [4] (r is the observation distance, d the aperture diameter, and λ the optical wavelength). The reason they provide is that the width of the *entire* electron beam in the experiment in Ref. [4] is in the range of $d = w_{\text{beam}} = 0.3$ to 2 mm, implying that in order to be in the far field, r must exceed 0.15 and 6.7 m, respectively, wherein the radiation was observed at a distance of approximately $r = 0.25$ m.

Here we explain why the experiment was indeed performed in the far field. The relevant parameter for calculating the interference of light emitted from a current source is the transverse coherence length–denoted by l_c–of the source, and not its total incoherent width (w_{beam}) as was assumed in the calculations of Ref. [1] when claiming against the measurements in Ref. [4]. The incoherent width can be thought of as a statistical distribution, where each arriving electron appears at a different location. Each single electron therefore consists of an independent wave function coherent only with itself, on a transverse length scale l_c. The coherent radiation emitted by each single electron, therefore, could only depend on a scale of up to l_c, and not w_{beam}.

This means that the correct effective aperture d from which the light is emitted in the case of a semiclassical calculation (as Ref. [1] employs) would be on the order of the transverse coherence length, i.e., $d = l_c$ and *not* $d = w_{\text{beam}}$, such that the Fraunhofer condition reads

$$r \gg l_c^2/\lambda.$$

In the experiment reported in Ref. [4], the transverse coherence length of the electron was measured to be $l_c = 5\ \mu$m for $w_{\text{beam}} = 0.3$ mm and $l_c = 33\ \mu$m for $w_{\text{beam}} = 2$ mm. Consequently, the Fraunhofer condition is easily met by the measurements, giving $r \gg 50\ \mu$m and $r \gg 2.2$ mm, respectively, where the distance in the experiment reported in Ref. [4] was $r = 0.25$ m. This implies that Ref. [4] safely meets the far-field condition. We further note that to avoid *geometrical* influence on the measurement (e.g., a parallax) one needs to ensure that the region of the different sources (i.e., the beam width w_{beam}) spans a small angle in the observation plane, which is also easily met by the experiment since $w_{\text{beam}}/r \ll 1$.

We complement these arguments with a numerical FDTD simulation of semiclassical Smith-Purcell emission by free electrons passing near a metallic grating, realizing the semiclassical equation (3) (see derivation in Appendix A). For numerical convergence purposes, we scaled down the problem to scanning electron microscope (SEM) energies ($E = 30$ keV) such that the grating length was $L = 4\ \mu$m with a periodicity of $\Lambda = 200$ nm for visible emission at $\lambda = 600$ nm for $\theta = 90^\circ$, and an incoherent beam width of $w_{\text{beam}} = 20\ \mu$m. We varied the coherence length l_c between 0.2, 1, and 4 μm, adding *incoherently* the contributions to the emitted light from different parts of the electron beam (i.e., adding incoherently 100 simulations with $l_c = 0.2\ \mu$m, 20 simulations with $l_c = 1\ \mu$m, etc.)—see Fig. 1. We propagated the near field to a distance $r = 100\ \mu$m. This distance is *larger* than the Fraunhofer distance associated with the coherence lengths ($r \gg l_c^2/\lambda$, with l_c^2/λ found to be at most 27 μm for $l_c = 4\ \mu$m), while being *smaller* than the Fraunhofer distance associated with the incoherent beam width w_{beam}, i.e., $r \ll w_{\text{beam}}^2/\lambda = 667\ \mu$m, as falsely argued by Ref. [1]. We find that, as expected, the divergence varies dramatically according to the coherence length l_c, even though the beam width w_{beam} stayed the same.

C. Ruling out the semiclassical theory

The calculation of spontaneous emission under the semiclassical interpretation considers a coherent interference from an extended source defined by $e|\psi(\mathbf{r})|^2$, where the width of $\psi(\mathbf{r})$ is of the order of l_c—thus defining the effective aperture $d = l_c$. At the same time, there is also an incoherent summation of the field intensities emitted from different parts of the electron beam, as the incoherent width of the electron beam w_{beam} is much wider than l_c. Under this interpretation, a change in l_c (between 5 and 33 μm) should, in the far field, lead to a respective change in the azimuthal divergence $\Delta\phi \sim \lambda/l_c$. Should no such change be observed, the semiclassical interpretation can be ruled out in favor of the incoherent pointlike emission, the quantum interpretation, predicted by quantum electrodynamics.

The experiment in Ref. [4] did not observe any change in the measured divergence, even though the coherence length l_c was varied by more than 6 times. Further, the observed divergence was more than an order of magnitude larger than that predicted by the semiclassical theory. This evidence therefore implies that *we can reject the semiclassical interpretation in favor of the quantum interpretation*, suggesting an invariant divergence.

VII. MODELING OF SPONTANEOUS EMISSION BY FREE ELECTRONS: THE ROLE OF ELECTRON POSTSELECTION

The key point is that Ref. [1] derives emission rates that depend on the final electron quantum state. This derivation corresponds to a postselection of the electron (or coincidence measurement of the electron and photon), which does not happen in the experimental setup of Ref. [4]. Such a postselection is experimentally feasible [10] and can be relevant to new experimental regimes of cathodoluminescence—wherein, the predictions presented in Ref. [1] can be of great importance. However, this is not the common situation in free-electron radiation (e.g., see the literature on cathodoluminescence

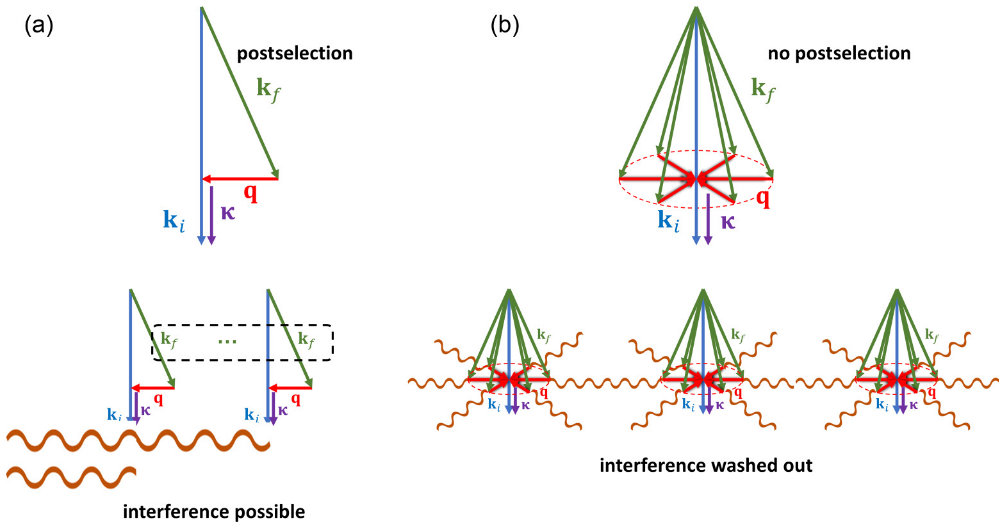

FIG. 2. Illustration of light emission by postselected and traced-out electron final states. (a) Free-electron emission from an initial electron momentum state $\mathbf{k}_i$ that scatters due to the emission into a postselected final state $\mathbf{k}_f$, where the emitted light has a wave vector $\mathbf{q}$, and the emission is mediated by the grating momentum κ. Momentum and energy conservation impose entanglement between the electron and photon. Postselecting electron final states allows the light emitted from different initial states of the wave function to interfere, making the emission wave function dependent. (b) When the electron final state is not measured, the entanglement between all possible final states and all possible photonic modes leads to quantum decoherence. In this case, the emission is wave function independent; i.e., light interference from different initial states of the electron wave function is washed out.

[11,12]), where only the light is measured, nor was it the case in the experiment in Ref. [4].

In Appendix B, we derive the emitted power spectrum in this manner, where we trace out the electron degrees of freedom by summing incoherently over all final momentum-space electron states $e^{i\mathbf{k}_\mathrm{f}\cdot\mathbf{r}}$. However, in Ref. [1] this incoherent summation is not performed. Instead, the paper mentions that the situation corresponding to not measuring the electron is analogous to specifying the final state of the electron to be a plane wave [1]. However, specifying the final electron state as a plane wave $e^{i\mathbf{k}_\mathrm{f}\cdot\mathbf{r}}$ means that it is postselected (with momentum $\mathbf{k}_\mathrm{f}$). This can be done by a momentum-resolved electron energy loss spectroscopy (EELS), but that did not happen in the experiment in Ref. [4].

We note that there is now strong evidence [2,4,13] that in the case of paraxial and perturbative free-electron radiation from transversely uniform media, with no electron postselection, the radiation does not depend on the initial electron wave function, due to the underlying electron-photon entanglement leading to decoherence [14,15]. Figure 2 illustrates this principle by showing that for light emitted by a coherent wave function, interference is washed out if the electron final state is not observed (i.e., no postselection of the final state is made).

It is important to also emphasize that there are possible exceptions to this rule in special situations in which the electron wave function does matter in spontaneous emission processes—in which the predictions of Ref. [1] can be valid, for example, nonparaxial electron wave packets (as thoroughly investigated in Ref. [1]). Another example is when the medium is not transversely uniform, and mediates scattering of initial electron states into a common final state [16]. This can be also understood from the overlap integral [Eqs. (4) and (5)] which will now depend on the probing of the local density of states by $|\psi|^2$. Other scenarios include nonperturbative interactions, in which the electron shape can affect the quantum state of light [17]; when measuring optical coherence of the emitted cathodoluminescence (CL) [18]; or when quantum recoil corrections [2,19] are dominant.

In this context, it is important to mention that our criticism does not mean that the calculations presented in Ref. [1] are not physical. Instead, they correspond to a different physical scenario where the electron is postselected. Furthermore, these derivations reveal new kinds of quantum corrections that are relevant in the sought-after regime of coincident electron-photon detection, beyond the paraxial approximation.

VIII. CONCLUSIONS

We lift the concerns regarding the experimental parameters used in Ref. [4], raised by the authors of Ref. [1]. We show both by simple optical arguments as well as numerical simulations that the measurements reported in Ref. [4] took place in the far field with respect to the electron transverse coherence length. From these measurements [4], we conclude that it is possible to reject the semiclassical theory in favor of the quantum theory.

We further suggest the authors clarify that the derivations following Eq. (10) in their paper [1] correspond to an experimental situation where the electron final state (after emission of light) is postselected (or measured in coincidence with the light). This is a crucial detail, since in this case the radiation will be dependent on the electron initial wave function [2,20], but the physical scenario is completely different. The common situation for measuring cathodoluminescence involves only measuring the light (as was done in the experiment of Ref. [4]), which corresponds to tracing out the electron final states [21]. This last step is what allows for the incoherent summation over many "point electrons" emitting light—the quantum interpretation—to be correct.

Upon lifting the concerns of Ref. [1], and explicitly noting the key requirement for electron postselection, we believe that Ref. [1] may constitute a valuable contribution to the problem of spontaneous emission by shaped electron wave functions. Specifically, the novelty of Ref. [1] is in deriving new expressions for the emission rates beyond the ubiquitous paraxial approximation and in showing that such rates could depend on the wave function upon postselection.

APPENDIX A: DERIVATION OF EQ. (3)

The semiclassical current density of an electron beam, for N electrons in different locations $\mathbf{r}_i$ in the beam, reads as

$$\mathbf{j}(\mathbf{r},t) = e\mathbf{v}_0 \sum_{i=1}^{N} |\psi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2 \delta(z - z_i - v_0 t).$$

From Jackson [5],

$$\frac{d^2P}{d\Omega d\omega} = \frac{\omega^2}{4\pi^2c^2}\left|\int dt \int d^3\mathbf{r}\hat{\mathbf{n}} \times [\hat{\mathbf{n}} \times \mathbf{j}(\mathbf{r},t)]e^{i\omega(t-\hat{\mathbf{n}}\cdot\mathbf{r}/c)}\right|^2,$$

giving

$$\frac{d^2P}{d\Omega d\omega} = \left|\sum_{i=1}^{N} e^{-i\omega\frac{z_i}{v_0}} \int d^2\mathbf{r}_T e^{-i\mathbf{q}_T\cdot\mathbf{r}_T} |\psi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2\right|^2 \times \left(\frac{d^2P}{d\Omega d\omega}\right)_{\text{class}},$$

since, for a single point particle $|\psi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2 \to \delta(\mathbf{r}_T - \mathbf{r}_{Ti})$ with $N = 1$ the term in the absolute value simplifies to 1.

For $N > 1$ we expand the absolute value squared:

$$\begin{aligned}\left|\sum_{i=1}^{N} e^{-i\omega\frac{z_i}{v_0}} \int d^2\mathbf{r}_T e^{-i\mathbf{q}_T\cdot\mathbf{r}_T} |\psi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2\right|^2 &= \sum_{i=1}^{N}\left|\int d^2\mathbf{r}_T |\psi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2 e^{-i\mathbf{q}_T\cdot\mathbf{r}_T}\right|^2 \\ &+ \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} e^{\frac{i(z_j-z_i)\omega}{v_0}} \int d^2\mathbf{r}_T |\psi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2 e^{-i\mathbf{q}_T\cdot\mathbf{r}_T} \\ &\times \int d^2\mathbf{r}'_T |\psi_T(\mathbf{r}'_T - \mathbf{r}_{Tj})|^2 e^{i\mathbf{q}_T\cdot\mathbf{r}'_T}.\end{aligned}$$

Now, for a general beam comprising N particles of random positions one needs to take the ensemble average. For dc currents below the critical current, following the same arguments of Appendix C the second term vanishes and we are left with the first term only:

$$\left\langle\left|\sum_{i=1}^{N} e^{-i\omega\frac{z_i}{v_0}} \int d^2\mathbf{r}_T e^{-i\mathbf{q}_T\cdot\mathbf{r}_T} |\psi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2\right|^2\right\rangle = \sum_{i=1}^{N} p(\mathbf{r}_i)\left|\int d^2\mathbf{r}_T |\psi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2 e^{-i\mathbf{q}_T\cdot\mathbf{r}_T}\right|^2,$$

where $p(\mathbf{r}_i)$ is the (classical) probability for an electron position at $\mathbf{r}_i$, such that for a continuous distribution of electron positions,

$$\left\langle\left|\sum_{i=1}^{N} e^{-i\omega\frac{z_i}{v_0}} \int d^2\mathbf{r}_T e^{-i\mathbf{q}_T\cdot\mathbf{r}_T} |\psi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2\right|^2\right\rangle = N\int d^2\mathbf{r}_{T0}p(\mathbf{r}_{T0})\left|\int d^2\mathbf{r}_T |\psi_T(\mathbf{r}_T - \mathbf{r}_{T0})|^2 e^{-i\mathbf{q}_T\cdot\mathbf{r}_T}\right|^2.$$

Giving the semiclassical prediction, Eq. (10),

$$\left(\frac{d^2P}{d\Omega d\omega}\right)_{\text{semi}} = \int d^2\mathbf{r}_{T0}p(\mathbf{r}_{T0}) \times \left|\int d^2\mathbf{r}_T |\psi_T(\mathbf{r}_T - \mathbf{r}_{T0})|^2 e^{-i\mathbf{q}_T\cdot\mathbf{r}_T}\right|^2 \times \left(\frac{d^2P}{d\Omega d\omega}\right)_{\text{class}}.$$

APPENDIX B: DERIVATION OF EQS. (4) and (5)

We would like to show that this result is general and not only restricted to Smith-Purcell radiation. We consider a relativistic transition current with respect to the interaction Hamiltonian $H_{\text{int}} = ec\boldsymbol{\alpha}\cdot\mathbf{p}$,

$$\mathbf{j}_{\text{fi}}(\mathbf{r},\omega) = ec\delta\left(\omega - \frac{E_i - E_f}{\hbar}\right)\boldsymbol{\psi}_{\text{f}}^{\dagger}(\mathbf{r})\boldsymbol{\alpha}\boldsymbol{\psi}_{\text{i}}(\mathbf{r}),$$

where $\boldsymbol{\psi}_{\text{i}}(\mathbf{r})$ and $\boldsymbol{\psi}_{\text{f}}(\mathbf{r})$ are the initial and final spinor wave functions of the electron.

The electric field emitted by this transition current is

$$\mathbf{E}_{\text{fi}}(\mathbf{r},\omega) = i\mu_0\omega\int d^3\mathbf{r}'\mathbf{G}(\mathbf{r},\mathbf{r}',\omega)\mathbf{j}_{\text{fi}}(\mathbf{r},\omega),$$

where $\mathbf{G}(\mathbf{r},\mathbf{r}',\omega)$ is the dyadic Green's function of the medium. Note that $\mathbf{E}_{\text{fi}}(\mathbf{r},\omega)$ is the field amplitude for a specified final electron state.

The emitted power spectrum to the far field measured by a distant observer at $\mathbf{r} = r\hat{\mathbf{n}}(\Omega)$ is

$$\frac{d^2P}{d\Omega d\omega} = 2r^2\epsilon_0 c\frac{1}{2\pi T_{\text{int}}}\langle|\mathbf{E}(\mathbf{r},\omega)|^2\rangle.$$

To obtain the expectation value of the light intensity, *when only the light is measured*, one traces out over all possible electron final states:

$$\langle|\mathbf{E}(\mathbf{r},\omega)|^2\rangle = \sum_{\text{f}}|\mathbf{E}_{\text{fi}}(\mathbf{r},\omega)|^2 = \mu_0^2\omega^2\int d^3\mathbf{r}''\mathbf{G}^{\dagger}(\mathbf{r},\mathbf{r}'',\omega) \times \int d^3\mathbf{r}'\mathbf{G}(\mathbf{r},\mathbf{r}',\omega)\sum_{\text{f}}\mathbf{j}_{\text{fi}}^{\dagger}(\mathbf{r}'',\omega)\mathbf{j}_{\text{fi}}(\mathbf{r}',\omega).$$

In a recently published paper [15], we have shown that for a single paraxial electron with density matrix $\rho_{\text{el}}(\mathbf{r},\mathbf{r}')$ and small photon recoil we have

$$\sum_{\text{f}}\mathbf{j}_{\text{fi}}^{\dagger}(\mathbf{r}'',t')\mathbf{j}_{\text{fi}}(\mathbf{r}',t) = e^2\mathbf{v}_0\ \mathbf{v}_0\ \delta(\mathbf{r}' - \mathbf{v}_0 t - \mathbf{r}'' + \mathbf{v}_0 t')\rho_{\text{el}}(\mathbf{r}' - \mathbf{v}_0 t, \mathbf{r}' - \mathbf{v}_0 t),$$

which, in the frequency domain, amounts to

$$\sum_{\mathrm{f}} \mathbf{j}_{\mathrm{fi}}^{\dagger}(\mathbf{r}'', \omega)\mathbf{j}_{\mathrm{fi}}(\mathbf{r}', \omega) = e^2 \hat{\mathbf{z}}\hat{\mathbf{z}} e^{i\omega \frac{z'-z''}{v_0}} \delta(\mathbf{r}'_T - \mathbf{r}''_T)\rho_{\mathrm{el},T}(\mathbf{r}'_T, \mathbf{r}'_T),$$

where $\rho_{\mathrm{el},T}(\mathbf{r}_T, \mathbf{r}'_T)$ is the transverse part of the density matrix. Therefore

$$\langle|\mathbf{E}(\mathbf{r}, \omega)|^2\rangle = \int d^2\mathbf{r}'_T \sum_{\alpha=x,y,z} e^2\mu_0^2\omega^2 \times \left|\int dz' G_{\alpha z}(\mathbf{r}, \mathbf{r}'_T, z', \omega) e^{i\omega\frac{z'}{v_0}}\right|^2 \rho_{\mathrm{el},T}(\mathbf{r}'_T, \mathbf{r}'_T),$$

and

$$\frac{d^2P}{d\Omega d\omega} = \int d^2\mathbf{r}'_T \left[2r^2\epsilon_0 c \frac{1}{2\pi T_{\mathrm{int}}} e^2\mu_0^2\omega^2 \sum_{\alpha=x,y,z} \times \left|\int dz' G_{\alpha z}(\mathbf{r}, \mathbf{r}'_T, z', \omega) e^{i\omega\frac{z'}{v_0}}\right|^2\right] \rho_{\mathrm{el},T}(\mathbf{r}'_T, \mathbf{r}'_T).$$

The expression in square brackets is the classical result, since it is obtained exactly when the electron is a point particle, giving finally

$$\frac{d^2P}{d\Omega d\omega} = \int d^2\mathbf{r}_T \frac{d^2P^{\mathrm{class}}}{d\Omega d\omega}(\mathbf{r}_T)\rho_{\mathrm{el},T}(\mathbf{r}_T, \mathbf{r}_T).$$

APPENDIX C: MANY-BODY EFFECTS USING SECOND QUANTIZATION

We again consider the emitted power spectrum to the far field measured by a distanced observer at $\mathbf{r} = r\hat{\mathbf{n}}(\Omega)$ to be

$$\frac{d^2P}{d\Omega d\omega} = 2r^2\epsilon_0 c \frac{1}{2\pi T_{\mathrm{int}}} \langle|\mathbf{E}(\mathbf{r}, \omega)|^2\rangle,$$

where

$$\langle|\mathbf{E}(\mathbf{r}, \omega)|^2\rangle = \mu_0^2\omega^2 \int d^3\mathbf{r}'' \mathbf{G}^{\dagger}(\mathbf{r}, \mathbf{r}'', \omega) \int d^3\mathbf{r}' \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \times \sum_{\mathrm{f}} \mathbf{j}_{\mathrm{fi}}^{\dagger}(\mathbf{r}'', \omega)\mathbf{j}_{\mathrm{fi}}(\mathbf{r}', \omega).$$

In another recently published paper [22], we have shown that, using second quantization [promoting the wave function $\psi(\mathbf{r}, t)$ to an operator $\hat{\psi}(\mathbf{r}, t)$], we can write

$$\sum_{\mathrm{f}} \mathbf{j}_{\mathrm{fi}}^{\dagger}(\mathbf{x}')\mathbf{j}_{\mathrm{fi}}(\mathbf{x}) = e^2\mathbf{v}_0\,\mathbf{v}_0 \Bigg[\underbrace{\langle\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}^{\dagger}(\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}(\mathbf{x})\rangle}_{\text{pair correlation}} + \underbrace{\delta(\mathbf{x}-\mathbf{x}')\langle\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{x})\rangle}_{\text{incoherent single electron}}\Bigg],$$

where $\mathbf{x} = \mathbf{r} - \mathbf{v}_0 t$ and $\mathbf{x} = \mathbf{r}' - \mathbf{v}_0 t'$.

We define the beam state as

$$|\mathrm{beam}\rangle = \left|1_{\mathbf{r}_1} 1_{\mathbf{r}_2} \dots 1_{\mathbf{r}_N}\right\rangle = a_{\mathbf{r}_1}^{\dagger} a_{\mathbf{r}_2}^{\dagger} \dots a_{\mathbf{r}_N}^{\dagger}|0\rangle,$$

where

$$a_{\mathbf{r}_i} = \int d^3\mathbf{r}\varphi(\mathbf{r} - \mathbf{r}_i)\psi(\mathbf{r})$$

is a wave-packet operator centered at $\mathbf{r}_i$, and to avoid complications due to Pauli exclusion, we assume $\{a_{\mathbf{r}_i}, a_{\mathbf{r}_j}^{\dagger}\} = \delta_{ij}$ [equivalent to no overlap between $\varphi(\mathbf{r} - \mathbf{r}_i)$ and $\varphi(\mathbf{r} - \mathbf{r}_j)$, which corresponds to the case of a large incoherent electron beam size with low transverse coherence, as in our experiment]. The expectation values are found,

$$\langle\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{x})\rangle = \sum_i |\varphi(\mathbf{x} - \mathbf{r}_i)|^2,$$

$$\langle\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}^{\dagger}(\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}(\mathbf{x})\rangle = \sum_i \sum_{j\neq i} |\varphi(\mathbf{x} - \mathbf{r}_i)|^2 |\varphi(\mathbf{x}' - \mathbf{r}_j)|^2,$$

which, in the frequency domain, amounts to

$$\sum_{\mathrm{f}} \mathbf{j}_{\mathrm{fi}}^{\dagger}(\mathbf{r}', \omega)\mathbf{j}_{\mathrm{fi}}(\mathbf{r}, \omega) = e^2\hat{\mathbf{z}}\hat{\mathbf{z}} e^{i\omega\frac{z-z'}{v_0}} \sum_i \sum_{j\neq i} |\varphi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2 |\varphi_T(\mathbf{r}'_T - \mathbf{r}_{Tj})|^2 \times e^{-i\frac{\omega}{v_0}(z_i - z_j)} + e^2\hat{\mathbf{z}}\hat{\mathbf{z}} e^{i\omega\frac{z-z'}{v_0}} \delta(\mathbf{r}_T - \mathbf{r}'_T) \sum_i |\varphi_T(\mathbf{r}_T - \mathbf{r}_{Ti})|^2,$$

where φ_T is the transverse part of the wave function.

We exemplify the many-body effects using Cherenkov radiation, for which the Green function simplifies

$$\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) = \frac{e^{in\frac{\omega}{c}r}}{4\pi r}(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}) e^{-i\frac{n\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{r}'}.$$

Substituting to the above equation we find

$$\frac{d^2P}{d\Omega d\omega} = \frac{\hbar\omega\alpha\beta}{2\pi}\sin^2\theta\,\delta\left(\cos\theta - \frac{1}{n\beta}\right)[N + N(N-1)S(\omega, \hat{\mathbf{n}})],$$

where we define the structure factor of the beam:

$$S(\mathbf{q}) = \left|\int d^2\mathbf{r}_T e^{-i\mathbf{q}_T\cdot\mathbf{r}_T} |\varphi_T(\mathbf{r}_T)|^2\right|^2 \frac{1}{N(N-1)} \sum_i \sum_{j\neq i} e^{-i\mathbf{q}\cdot(\mathbf{r}_i - \mathbf{r}_j)}.$$

Note that

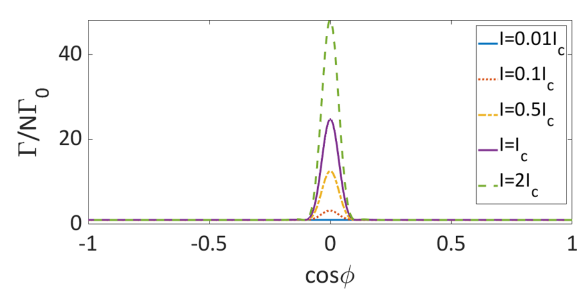

FIG. 3. Numerical simulation of the normalized emission rate from a beam of N electrons as a function of the azimuthal angle $\cos\phi$. The beam current is varied between a value much lower than the critical current I_0 to a higher value, resulting in a superradiant effect. Each simulation is carried out with 100 repetitions. For values much lower than I_0 (as in Ref. [4]), the emission is azimuthally uniform.

(1) For point electrons, $|\varphi_T(\mathbf{r}_T)|^2 \to \delta(\mathbf{r}-\mathbf{r}_T)$, this formula converges to the classical result [9].

(2) For dc currents, the electron wave packets are randomly positioned in the beam.

We can write the emission rate simply as

$$\Gamma = N\Gamma_0[1 + (N-1)S(\mathbf{q})],$$

where Γ_0 is the single-electron emission rate. The number of electrons in the beam, for a given current, total (incoherent) beam length L, and electron velocity v is

$$N = \frac{IL}{ev}.$$

The critical number of electrons in the beam N_c above which many-body correlations are important, is the one for which the area per electron becomes comparable to the wavelength squared. If W is the total (incoherent) beam width, then the average area per single electron is WL/N, and comparing this value to the emitted wave vector we obtain

$$N_c = q^2 WL.$$

Note that

$$\frac{N}{N_c} = \frac{\lambda^2}{4\pi^2 Wev} I = \frac{I}{I_0},$$

where

$$I_0 = q^2 Wev$$

is the critical current.

To show this, we numerically compute the structure factor using Monte Carlo simulations,

$$\langle S(\mathbf{q})\rangle = \frac{1}{N(N-1)}\left\langle \sum_i \sum_{j\neq i} e^{-iq\cos\theta(x_i-x_j)\cos\phi} e^{-i\frac{\omega}{v}(z_i-z_j)}\right\rangle,$$

for Cherenkov radiation, and plot the value of $\Gamma/N\Gamma_0$ as a function of $\cos\phi$ for variable currents I/I_0. See Fig. 3.

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