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Solitonets: complex networks of interacting fields

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We propose complex networks made with interacting fields, where the interaction dynamics at each individual node in the system has infinite degrees of freedom. We construct networks, based on the interactions between vector solitons, whose dynamics are governed by conservation laws. Hence, the dimensionality of the dynamics at each node is determined by the initial conditions, making the problem tractable. We present examples of small and large soliton-based networks, and demonstrate memory effects within them that are enormously enhanced by noise. Finally, we demonstrate that such networks, with infinite-dimensional dynamics, can exhibit spontaneous self-synchronization effects.

Keywords: complex networks solitons; nonlinear dynamics; complex systems; self-synchronization; nonlinear optics

Complex networks have been fascinating scientists for more than a decade. They appear in diverse areas, ranging from natural systems such as food chains and metabolic networks, to man-made systems such as electrical power grids and the Internet (see Barabási & Albert (1999), Strogatz (2001) and Albert & Barabási (2002) for reviews on complex networks). At the heart of the complex networks lies the fact that a complex nonlinear system does not behave as a superposition of its building blocks. Rather, a network can display dynamics of its own, either collective, where the network behaves as one entity, partially collective or sometimes highly fragmented, with different parts of it behaving in a completely uncorrelated fashion. The internal dynamics of the complex networks can take on many forms, some conceived as crucial to the existence of life (e.g. DNA repair systems; Kohn 1999), some posing hazards to civilization (spread of infectious diseases; Wallinga *et al.* 1999) and some entertaining, linking seemingly unrelated events (Kirby & Sahre 1998; Strogatz 2001). Much of the work on the complex networks concentrates on their structure, and does not address the evolution dynamics of the signals within the networks (Barabási & Albert 1999; Albert & Barabási 2002). However, most realistic networks have complex

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internal dynamics, which is actually very interesting in its own right (Strogatz 2001). The nonlinear dynamics exhibited in the complex networks gives rise to a wealth of fascinating phenomena, ranging from rather simple cases of fixed points and limit cycles, to more complex dynamics, such as strange attractors, all kinds of chaotic motions, self-synchronization of chaotic systems and much more. Whereas research on the structure of the complex networks has been extensive for years, studying the internal dynamics of the complex networks is relatively new. However, it is already established that the internal dynamics of the complex networks plays a crucial role in the structural formation of the network itself, as happens even in natural web-like structures (Ito & Kaneko 2002). As such, the internal dynamics of complex networks is now naturally attracting more and more research interest. Perhaps the most important challenge in the current research on the complex networks is the one defined by Wilson (1998) and highlighted by Strogatz in his keynote review (Strogatz 2001): 'The greatest challenge today … is the accurate and complete description of complex systems.'

Here, we take the next step in this vision, and propose complex networks constructed from interacting fields, where the interaction dynamics at each individual node in the system has infinite degrees of freedom. We use solitons as the 'carriers of interactions' between nodes in the network. In doing that, we take advantage of the generic properties of solitons (Segev & Stegeman 1998; Stegeman & Segev 1999), and construct networks made of interacting fields, in which the dynamical parameter characterizing the internal evolution is the ratio between the amplitudes of the fields comprising the solitons, while all other properties (number of solitons, power and momentum they carry, etc.) are conserved. Since the solitons are fields, the number of degrees of freedom for the interaction at each node in the network (the 'interaction dimension') is, in principle, infinite, which could render the problem intractable. However, the conservation laws of solitons imply that the number of different solitons propagating within the network is uniquely defined by the initial conditions. This feature is what makes the problem tractable. As examples, we study the networks within which trains (sequences) of two vector solitons propagate and interact with one another at every node. We study memory effects in such soliton-based networks, and show that memory is enormously enhanced by noise. We also demonstrate that such a network, with infinite-dimensional dynamics, can exhibit spontaneous self-synchronization effects.

Let us discuss first the dimensionality of the nonlinear interaction at each node of the network, in terms of the possible degrees of freedom. In complex networks studied thus far, the number of available degrees of freedom for the interaction is limited, and is typically very small. Examples range from the simplest case of binary interactions (bits), to other kinds of interactions in man-made networks and in network simulations (Strogatz 2001). In all of those previously studied networks, the interaction at each node is mapping a finite number of states onto itself. For example, if the interaction maps a 0 or 1 state onto a 0 or 1 state, the mapping is done by a 2×2 table. Alternatively, if the interaction maps a vector of length N containing binary states on a vector of the same structure, the table would be $2^N\times2^N$. Hence, if the network contains M nodes, the number of available states in the network would be $(2^N)^M$, which is a finite number. The next stage of complexity can be found in physical networks (Ito & Kaneko 2002), where the state at each node is continuous, and hence is described by one (or more) real (or complex) number. Here, each node has an infinite number of possible states, but the dynamics at each node is completely described by a single point in the limited dimensional phase space. For example, consider a particle in a box, whose state is described by the position and velocity in a three-dimensional space. The state of such a particle is fully described by six real numbers (three for position and three for velocity). Hence, the dynamics of a network constructed from M nodes, with N degrees of freedom for each node ($N=6$ for the particle example), is described by a point in the combined phase space of $N\times M$ dimensions.

In sharp contrast, the networks we study here have an infinite-dimensional interaction at each and every node of the network, because the interaction at each node is between continuous fields. The interaction at each node maps a state in an infinite-dimensional, continuous, phase space onto another state in the same phase space. Consequently, the number of degrees of freedom for the interaction at each node is infinite. In this sense, for a network made of the interacting fields, the (infinite) phase space of a single node is larger than that of any entire network studied previously (e.g. the $N\times M$-dimensional phase space in a network made of a 'particle in a box' at every node). In a network relying on the interacting fields, increasing the number of nodes by one adds another infinite-dimensional phase space to the dynamical system. The difference between a network made of the interacting fields and other networks is therefore profound.

In principle, a field-based network whose phase space is infinite is most likely to be theoretically intractable, because the dynamics, even at a single node, is described by a field whose dimensionality is infinite. However, we use solitons as the carrier of interactions. Solitons, self-localized wave packets, which behave and interact with one another as real particles do (Segev & Stegeman 1998; Stegeman & Segev 1999), obey conservation laws. In particular, for integrable systems, the number of conservation laws is infinite (Ablowitz & Segur 1981), which, as we show below, is exactly what makes the soliton-based networks tractable. A natural (simplest) choice for using solitons in the networks could be the solitons of the cubic nonlinear Schrodinger equation (Kerr solitons). However, such solitons have been studied for computation purposes, and it was proven that the interactions between them are 'oblivious', in terms of information processing (i.e. computation machines made up from Kerr solitons would never be Turing equivalent; Jakubowski *et al.* 1997). On the other hand, the Manakov solitons (Manakov 1974) comprised of two interacting fields, have been shown to be able to perform Turing-equivalent computations (Steiglitz 2001*a*,*b*; Rand *et al.* 2005). For this reason, we propose complex networks constructed from the Manakov solitons. Furthermore, as shown by Steiglitz (Jakubowski *et al.* 1998), the interaction between a pair of Manakov solitons can be expressed as a simple bilinear transformation between complex numbers, which actually makes these simulations of large field-based networks possible (otherwise, with non-soliton fields, the interaction at each node of the network would require solving a dynamic nonlinear PDE; this would render simulations of large networks unmanageable). The interaction between the Manakov solitons at a given node is shown in figure 1. The variables x, a, y and b are complex numbers representing the ratios between the fields comprising the 'dark grey' and 'light grey' Manakov solitons at the input (x and a) and output (y and b) of the node. The interaction (see the formula in figure 1) in essence maps two complex numbers onto another two complex numbers.

nonlinear interaction element

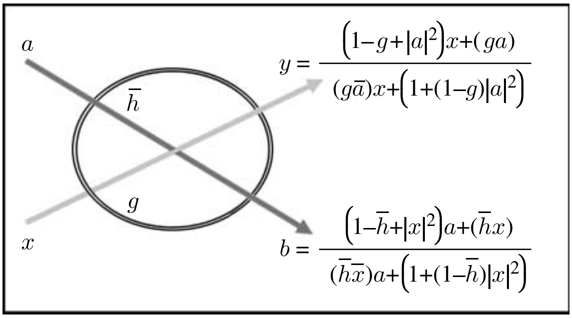

Figure 1. Illustration of a node in the network and how it operates, based on the interaction between two Manakov solitons. The solitons are marked by the dark and light grey arrows, while the interaction region is indicated by the black circle. The parameters defining the conserved quantities of the dark and light grey solitons (power and velocity) are contained in the complex numbers g and $\bar{h}$. The input states of the solitons are a and x, whereas the output states are b and y, for the dark and light grey solitons, respectively. The formula provides the bilinear transformation describing the interaction, mapping the input states onto the output states (Jakubowski *et al.* 1998).

One of the consequences of the integrability of the Manakov solitons is that the number of solitons is conserved, and so is the power and momentum of each one of them (Ablowitz & Segur 1981). This means that, in a network, the kinds of the Manakov solitons circulating in the network are uniquely defined by the stream of incoming solitons into the network. The minimum number would be one, having solitons with identical parameters (power and velocity) circulating in the network. In this case, however, the solitons never collide, but rather propagate in parallel with one another. The next option is launching two kinds of Manakov solitons into the network, where the simplest case is two solitons of identical power but different velocities. This is exactly what we do here. Figure 2*a*,*b* depicts a small and large network into which two kinds of Manakov solitons are launched: the light grey soliton is circulating in the system, whereas the dark grey soliton enters at one node and leaves at another. We emphasize that the quantity circulating in these networks is just the ratio between the field constituents of the solitons; the net number of solitons in the network at any time is fixed, and hence the power circulating in the network is constant in time. Each soliton is uniquely defined by its invariant features (powers and velocities), and by its dynamical features (ratio between the field constituents comprising each soliton), which can vary upon interaction. In the example shown in figure 1, the power and velocity are incorporated within the real and imaginary part of g (or $\bar{h}$) of the light grey (dark grey) soliton, whereas the dynamical state of each are x and a, which are mapped onto y and b, for the light and dark grey solitons, respectively. In other words, the state of each soliton is represented by a dimensionless complex number. As such, what circulates in the network, reflecting its complex internal dynamics, is information alone, representing the state of each soliton entering or leaving a node in the network. Such a network presents a simple connection between the dynamics and topology of a network. Because the chosen interaction mechanism restricts the network topology (figures 1 and 2), the degree of the input and output states at each node is exactly 2.

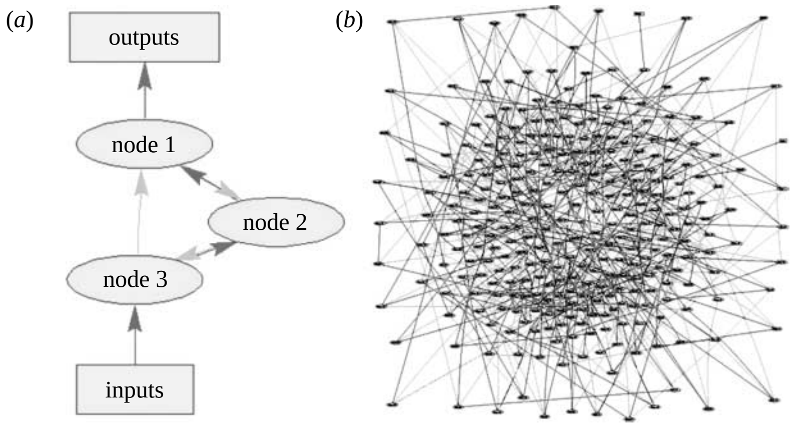

Figure 2. Examples of small and large soliton-based complex networks. (*a*) The small network is made of three interaction elements (nodes), an input element and an output element. This is, in fact, the smallest possible Manakov soliton network with a single input and output that exhibits a memory behaviour. (*b*) The large network consists of 300 interaction elements plus one input and one output element.

Having constructed soliton-based networks, the question is, what are they good for? It is yet unknown if soliton-based networks could provide a better means for information processing. Irrespective of the potential applications, we demonstrate here several unique features of soliton-based networks. Figure 3 shows noise-enhanced memory effects in small and large networks. The dark and light grey solitons at all nodes in both networks of figure 3 are initialized into a background state a long time before $t=0$. Then, at $t=0$, a sequence of identical solitons (i.e. $a=a(t)=a_0$) is launched into the network at some particular ('input') node, between $t=0$ and some other specific time t_{f}. These solitons circulate in the network, interacting with the background state and with one another, and we examine the amplitudes of the solitons remaining in the network after $t=t_{\mathrm{f}}$, i.e. we study memory effects.

Let us now explain the process in more detail. The ground state of the system occurs when all the solitons circulating in the network go through one another unaffected. This happens when all the solitons are at a 'background state' of equal fields: $x=a=1$. The physical meaning of such a background state is that the soliton power is divided evenly between the two fields comprising it, and their relative phase is zero. Consequently, when two such solitons (of $x=a=1$) collide, they go through each other unaffected ($y=b=1$). For this reason, we denote these as 'background solitons'. An example of such a background state, in a small network containing one input node, one output node and three interaction nodes, is shown in animation 1 in the electronic supplementary material. Then, after the system has been initialized into its background state, we launch through the input, starting at $t=0$, a sequence of 'charged solitons': solitons that are different from the background solitons, i.e. $x\neq 1$ and/or $a\neq 1$. These charged solitons replace the input background solitons of $t<0$, and are now launched into the network at equal intervals for $0\leq t\leq t_{\mathrm{f}}$. The charged solitons interact with the background solitons in a non-trivial way: their dynamic parameter changes in each interaction ('collision') at each node, at each clock interval (animation 2 in

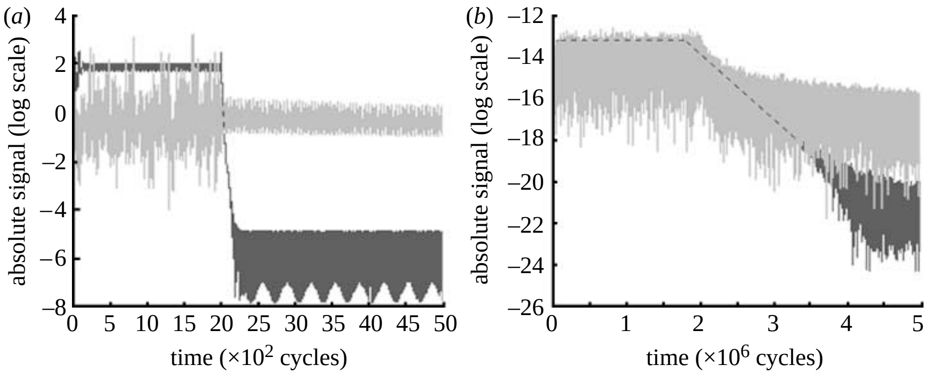

Figure 3. Noise-enhanced memory effects in soliton-based networks. Memory effects (dark grey curves) and their huge enhancement (light grey curves) occurring by superimposing stochastic noise on the input signals. Shown are the absolute values of the ratios between fields comprising the solitons at some node, as they evolve in time. The signal is defined as the difference between the output state and the 'background value', which was chosen to be 1. It is measured at the dark grey soliton output of a typical element. The outcome of a DC input signal is plotted in dark grey, while the outcome of a noisy input signal is plotted in light grey. (*a*) For the small network, the signal was turned 'on' for 2000 time cycles, and then the memory is measured for another 3000 time cycles. As long as the input signal is on, the DC is somewhat higher than the response to a noisy signal. However, after the input signal is terminated (and the input is switched back to a background value of 1), the memory left for the noisy signal, after the signal is stopped, is greatly enhanced—by several orders of magnitude—compared to the memory left after the DC signal. (*b*) For the large network, the signal is turned on for 20 000 time cycles, and then the memory is measured for another 30 000 time cycles. After a longer transient time, a steady-state memory is reached. Again, the memory for the noisy signal is larger than that for the DC signal by several orders of magnitude.

the electronic supplementary material). Then, at $t = t_{\mathrm{f}}$, the input sequence of charged solitons is terminated, and the input state is switched back to a sequence of background solitons ($a = 1$). However, the charged solitons continue to circulate in the network for a long time after $t = t_{\mathrm{f}}$, without decaying to the background state (dark grey curves, figure 3; animation 3 in the electronic supplementary material). This, in itself, is not surprising, because both of these networks contain closed loops, so it is expected that some 'memory' will survive from the time-limited stream of dark grey solitons. However, when we superimpose some small-amplitude stochastic noise (i.e. $a = a_0 + n(t)_{\mathrm{noise}}$, where $n(t)$ has a small mean value, $\langle n(t) \rangle \ll a_0$, and a small variance) on the input stream of charged solitons, the memory is enhanced by many orders of magnitude (dark grey curves). In the small network (figure 3*a*), the addition of the stochastic noise brings the memory to the level of the input signal. In the large network (figure 3*b*), the memory enhancement via noise is even larger (more than six orders of magnitude), when compared with the memory effects induced by a constant small amplitude signal ($a = a_0$).

We observe another unique feature of the soliton-based networks: spontaneous self-synchronization. When we launch a train of identical dark grey (or light grey) solitons into the networks, after some time, the entire network operates in a synchronous fashion, as if a clock is timing it (figure 4*a*–*c*). We find that this behaviour happens in both the small (figure 4*d*) and large (figure 4*e*) networks.

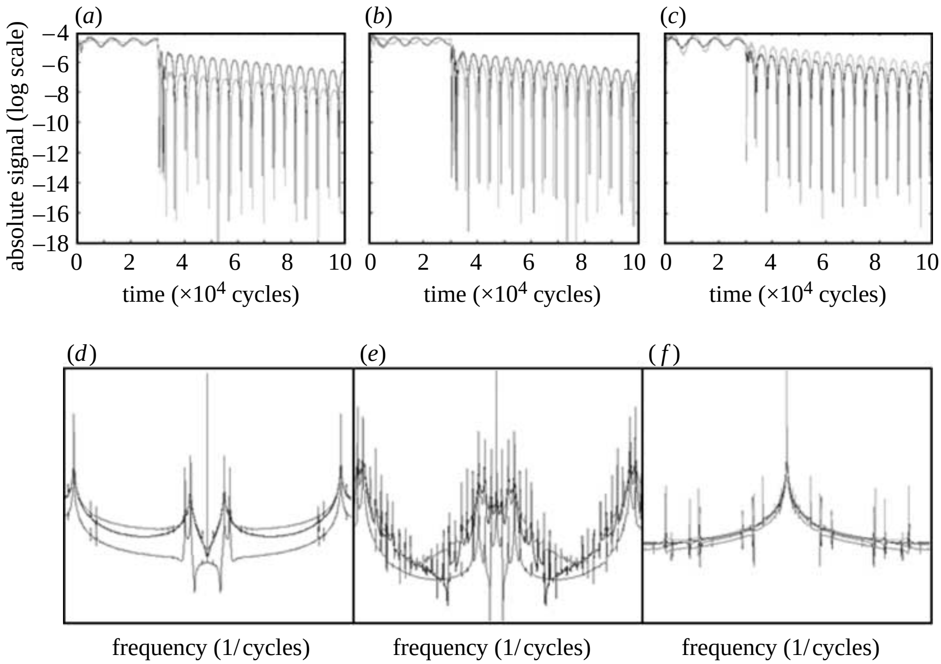

Figure 4. Self-synchronization effects in soliton-based networks. (*a–c*) The output signals of the dark and light grey solitons plotted for three distinct elements, in a large network exhibiting synchronization. (*d–f*) Spectral images of the memory presenting synchronization at a set of dominant frequencies. These spectra are created by Fourier transforming the memory signal in the light grey soliton output of several distinct elements. Synchronization appears as the common 'peaks' that are found exactly at the same frequencies for all elements in the network. (*d*) The spectrum of signals in a small network (three interaction elements), measured after the input DC signal was turned off. (*e*) The spectrum of signals in a large network (300 interaction elements), measured after the input DC signal was turned off. (*f*) The spectrum of signals in a small network (three interaction elements), with a noisy input signal. Even for the noisy signal, synchronization is found.

Moreover, we find that spontaneous self-synchronization occurs also when the train of input solitons is stochastic: an input stream of solitons with high variance (much greater than 1) can also induce a self-synchronization behaviour (figure 4*f*). Such self-synchronization of a network with a stochastic input is manifested in the appearance of dominant frequencies in the Fourier map of each node in the network; however, because the input is stochastic, the phases of different nodes are not synchronized. In other words, whereas a network with a regular input (a train of solitons) self-synchronizes and operates in unison, a network with a stochastic input displays dynamics at a common dominant frequency, but the phases at different nodes vary differently in time. What is even more surprising is that the dominant self-synchronized frequency depends on the network structure alone, and not on the input solitons or the initial conditions at each node.

Before closing, we would like to discuss the effects of nonlinearity in such soliton networks. Of course, the solitons are entirely nonlinear entities, so having nonlinearity is a prerequisite to constructing a soliton-based network.

Notwithstanding that, it is important to ask what role does the nonlinearity play in the network dynamics. To this extent, we compare the soliton-based networks studied here with a linearized network of the same structure. To do that, we linearize the expressions shown in figure 1 for the input signals close to unity ($x \approx 1$, $a \approx 1$), and repeat the network simulations. (Physically, linearization means that the field constituents of each soliton have almost the same power and phase.) We find that, in the linearized networks, the memory enhancement for noisy signals is considerably smaller and can be neglected. As for the synchronization, we find that, for a linearized network, as for any linear system, each frequency evolves independently of the others. That is, in the linearized version of the soliton networks, no dominant frequency ever develops, as opposed to the fully nonlinear networks where the information is transferred between frequencies, creating a dominant peak in the Fourier map. Altogether, both phenomena we have demonstrated, namely the noise-enhanced memory effects and the self-synchronization, are a direct outcome of having the networks operate nonlinearly.

To conclude, we have proposed soliton-based complex networks and demonstrated some of their unique features. The topology of the large complex network demonstrated here (figure 1*b*) was randomly generated, along with many other realizations of network topologies. However, all of these soliton-based complex networks exhibit the same behaviour, of noise-enhanced memory effects and self-synchronization, irrespective of the actual network topology. This implies that the unique features we identified for soliton-based networks are actually universal, almost unrelated to the specific network structure. Perhaps it is a little early to expect an experimental construction with a large soliton-based network, but certainly a small network based on the Manakov solitons (as shown in figure 2*a*) can be constructed today (Anastassiou *et al.* 1999, 2001; Rand *et al.* 2007). Such a network is expected to exhibit several unique features, such as noise-enhanced memory effects and self-synchronization (of states) to a common dominant frequency. Since that dominant frequency is independent of the input stream and the initial conditions at the nodes, it is interesting to ask what this dominant frequency tells us about the network structure and connectivity. Even more interesting questions actually arise when the network is operated with three (or more) kinds of solitons circulating in it. For example, a three-soliton network has inherently three types of nodes, one for each combination of 'colours'. Certainly, this would add much complexity to the networks, as the number of degrees of freedom at each node is uniquely defined by the number of kinds of solitons. Another option is to construct soliton-based networks where the interactions are not pairwise; for example, a network based on three solitons interacting simultaneously. In such soliton-based networks, the number of degrees of freedom at each node is uniquely defined by the number of solitons interacting and could be, in principle, infinite. Moreover, by controlling the number of conservation laws, it is possible to investigate the whole range of complexity between the infinite-dimensional fields and the finite-dimensional dynamics of solitons. The range of possibilities arising from the ideas presented here is large, and the dynamics of such networks most probably offers many more interesting features that cannot be thought of at this early stage. Last but not least, the ideas presented here should be viewed in the context of statistical mechanics: considering complex networks whose interaction carriers are fields is similar to ascribing a wave function to each (deterministic) particle, which has led to the quantum generalization of statistical mechanics. In this perspective, what we have presented here is a generalization of complex networks: networks constructed from interacting fields.

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