Abruptly Focusing and Defocusing Needles of Light and Closed-Form Electromagnetic Wavepackets

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Supplementary Information

Contents

1 Exact, closed-form vector solutions of electromagnetic pulses

The objective of this section is to give a more detailed discussion surrounding Eq. (4) of the main text. For ease of reference, and to make the Supplementary Information self-contained, we reproduce our solution here. Our solution, at cylindrical coordinates (*r*, *z*) and time *t*, can be expressed by the scalar function

$$\psi = \frac{\mathrm{c}t - \mathrm{i}a}{k_0 \tilde{R}^2}\left(\frac{1}{k_0 \tilde{R}} f^{-s-1} + \frac{s+1}{s} f^{-s-2}\right), \tag{S1.1}$$

where $f \equiv 1 - \mathrm{i}k_0/s\left(z + \mathrm{i}\tilde{R} - \mathrm{i}a\right)$, $\tilde{R} \equiv \left[r^2 + (\mathrm{ic}t + a)^2\right]^{1/2}$ is the complex length, $\omega_0 = k_0\mathrm{c}$ is the central angular frequency of the pulse, and c is the speed of light in the linear, homogeneous, time-invariant and isotropic medium (not necessarily free space). Parameters *a* and *s* control the focal spot size and pulse length of the electromagnetic pulse. In the limit of a weakly-focused, many-cycle pulse, *a* approximates the Rayleigh range of the beam, i.e., $a \approx k_0 w_0^2/2$, where w_0 is the beam waist radius (second irradiance moment) at the focal plane, and $s \approx \omega_0^2 \tau^2/(4\ln 2)$, where τ is the intensity full-width-at-half-maximum (FWHM) pulse duration.

In Section A1, we can show that $\mathrm{Re}\{f\} \geq 1, \forall r, z, t, a, s \in \Re, a, s > 0$ (details in Section A1), guaranteeing a complete absence of singularities in (S1.1) and thus a physically valid description at every point in space-time.

Equation (S1.1) exactly solves the electromagnetic wave equation (with azimuthal symmetry)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0. \tag{S1.2}$$

As discussed in the main text, vector solutions of electromagnetic fields **E** and **H** are readily obtained by treating the scalar solution (S1.1) as a component of Hertz vectors $\mathbf{\Pi}_{\mathrm{e}}$ and $\mathbf{\Pi}_{\mathrm{m}}$, and applying the equations [35]

$$\begin{aligned} \mathbf{E} &= \nabla\times\nabla\times\mathbf{\Pi}_{\mathrm{e}} - \mu\frac{\partial}{\partial t}\nabla\times\mathbf{\Pi}_{\mathrm{m}} \\ \mathbf{H} &= \nabla\times\nabla\times\mathbf{\Pi}_{\mathrm{m}} + \varepsilon\frac{\partial}{\partial t}\nabla\times\mathbf{\Pi}_{\mathrm{e}} \end{aligned}, \tag{S1.3}$$

where ε and μ are the medium's permittivity and permeability respectively.

For instance, the radially-polarized TM10 (Fig. S1.1) and azimuthally-polarized TE10 modes are obtained by setting $\mathbf{\Pi}_{\mathrm{e}} = \psi\hat{z}$, $\mathbf{\Pi}_{\mathrm{m}} = 0$, and $\mathbf{\Pi}_{\mathrm{e}} = 0$, $\mathbf{\Pi}_{\mathrm{m}} = \psi\hat{z}$ respectively. Linearly-polarized fundamental modes (Fig. S1.2) are obtained by setting $\mathbf{\Pi}_{\mathrm{e}} = \psi\hat{x}$, $\mathbf{\Pi}_{\mathrm{m}} = 0$, or $\mathbf{\Pi}_{\mathrm{e}} = 0$, $\mathbf{\Pi}_{\mathrm{m}} = \psi\hat{y}$, or some linear combination thereof. A spectral analysis, discussed in Section 3, also confirms that (S1.1) contains neither backward-propagating components nor DC (i.e., zero-frequency) components. In fact, (S1.1) has non-zero components only in the positive ω-k_z quadrant of the spectral domain, making our solutions the *first* finite-energy closed-form solutions of Maxwell's equations that is free of approximations. This is exciting because all existing closed-form models for ultrashort, non-paraxial pulses to-date suffer from drawbacks like the existence of points of divergence (where the field goes to infinity, which is non-physical) and backward propagating components [36-44].

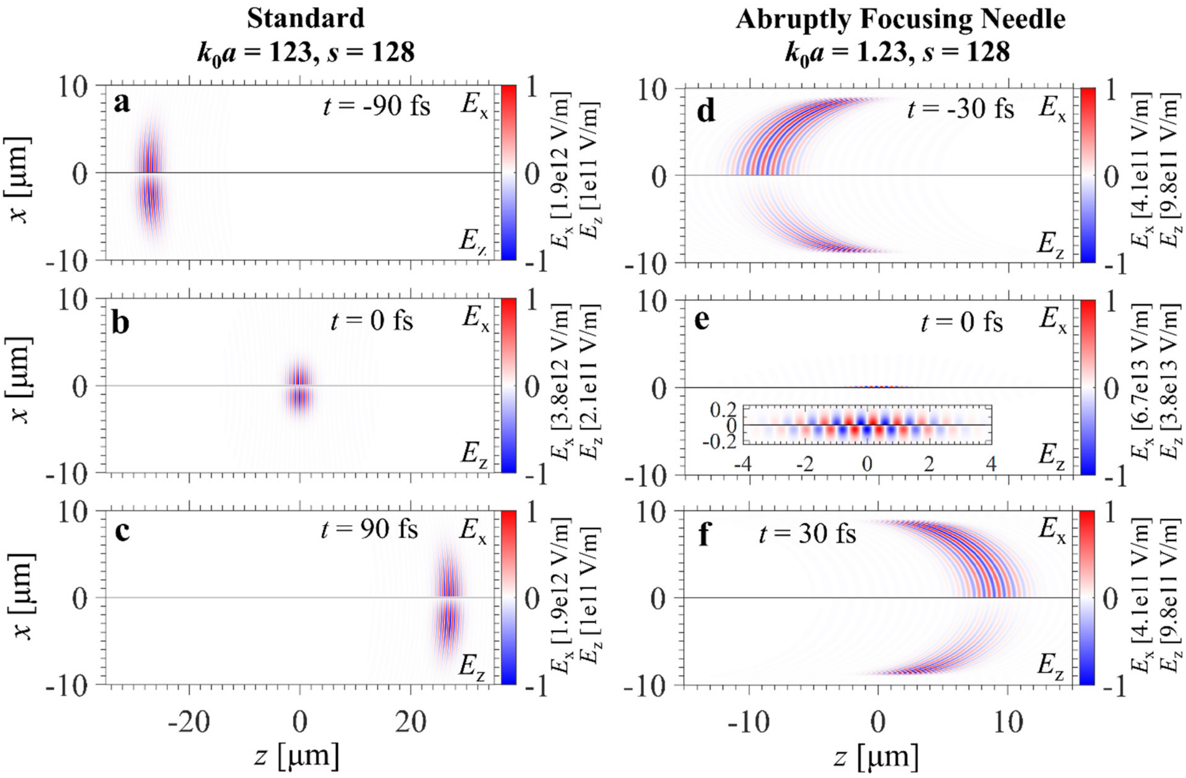

Figure S1.1. Fields of a linearly-polarized vectorial temporally diffracting (TD) electromagnetic pulse in the standard regime (a-c) and in the Abruptly Focusing Needle regime (d-f). In both cases, the total pulse energy is 1 mJ and the central wavelength is 0.8 μm. The field profile is shown in the y = 0 plane.

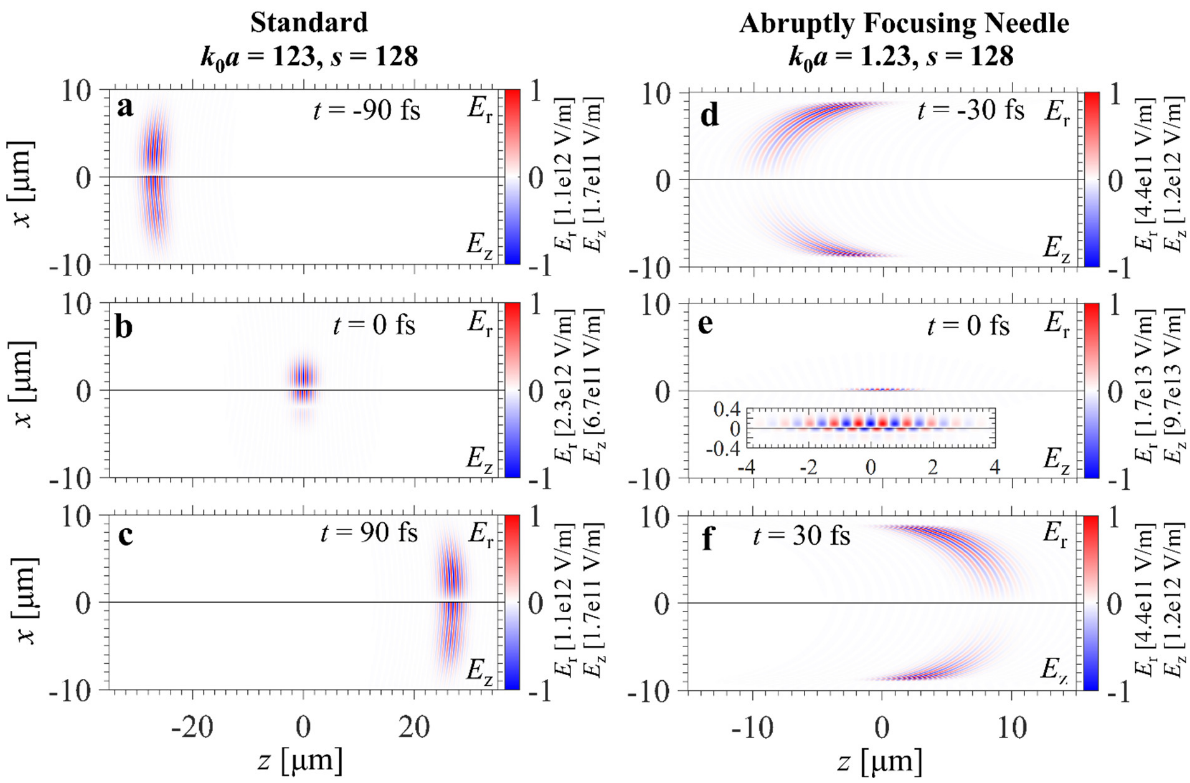

Figure S1.2. Fields of a radially-polarized (TM10) vectorial temporally diffracting (TD) electromagnetic pulse in the standard regime (a-c) and in the Abruptly Focusing Needle regime (d-f). In both cases, the total pulse energy is 1 mJ and the central wavelength is 0.8 μm. The field profile shown is azimuthally symmetric.

2 Derivation and properties of the electromagnetic wavepacket

In this section, we present the main steps of our derivation for (S1.1), which is also Eq. (4) of the main text. For a single frequency $\omega = k$c, the electromagnetic wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0\,. \tag{S2.1}$$

reduces to the Helmholtz equation

$$\frac{\partial^2 \psi'_0}{\partial z^2} = -\nabla_\perp^2 \psi'_0 - k^2 \psi'_0\,, \tag{S2.2}$$

where $\psi' = \psi'_0(x, y, z, \omega)\exp(-\mathrm{i}\omega t)$ solves Eq. (S2.1). Equation (S2.2) governs the beam evolution along the z direction. By the Fourier transform limit (Eq. (1) in the main text), if the intensity $|\psi|^2$ is highly localized in x, then a large spread in k_x must result. By Eq. (S2.2), this large spread causes the localization in x to be rapidly destroyed with propagation distance z.

To formulate a solution where the Fourier transform limit leads instead to a tradeoff between spatial narrowness and temporal duration – thus lifting the restriction on the spatial shape of the high intensity region at the focus – it stands to reason to reverse the roles of z and t (or of k_z and ω in the spectral domain). In other words, we begin with a wave comprising a single wavevector component k_z that evolves according to

$$\frac{\partial^2 \psi_0}{\partial t^2} = c^2\nabla_\perp^2 \psi_0 - c^2 k_z^2 \psi_0\,. \tag{S2.3}$$

where $\psi = \psi_0(x, y, t, k_z)\exp(\mathrm{i}k_z z)$.

A general solution to (S1.2) takes the form

$$\psi_1 = \int_{-\infty}^{\infty} \psi_0 F(k_z) \mathrm{e}^{\mathrm{i}k_z z} \mathrm{e}^{k_z a} \mathrm{d}k_z \,, \tag{S2.4}$$

where ψ_0 satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_0}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 \psi_0}{\partial t^2} - k_z^2 \psi_0 = 0 \,, \tag{S2.5}$$

which is the azimuthally symmetric version of (S2.3). A solution to (S2.5) is

$$\psi_0 = \frac{\exp\left(-k_z \widetilde{R}\right)}{k_0 \widetilde{R}} \,, \tag{S2.6}$$

where $\widetilde{R} \equiv \left[r^2 + (\mathrm{i}ct + a)^2\right]^{1/2}$ and k_0 is a normalizing constant. By choosing the branch of the square root in $\widetilde{R}$ such that $\mathrm{Re}\{\widetilde{R}\} > 0$, we can show that $\mathrm{Re}\{\widetilde{R}\} \geq a, \forall r, t, a \in \Re, a > 0$ (details in Section A1). In addition, we will build our solution only from components where $k_z \geq 0$. Under these conditions, one can see that (S2.6) contains absolutely no singularities and is a physically valid solution at all points in space-time.

Equation (S2.4) is a valid solution of (S1.2) for an arbitrary spectral function $F(k_z)$. To obtain (S1.1), we choose a Poisson distribution that peaks at $k_z = k_0$

$$F(k_z) = \left(\frac{s}{k_0}\right)^{s+1} \frac{k_z^{\mathrm{s}} \exp(-s\,k_z/k_0)}{\Gamma(\mathrm{s}+1)} \theta(k_z) \,, \tag{S2.7}$$

where Γ and θ are the Gamma and Heaviside functions respectively. The choice of a Poisson distribution is partly motivated by the fact that – coupled with the free space photon dispersion relation which requires $k_z = 0$ when $\omega = 0$ – it leads to a solution that contains no DC (i.e. zero-frequency) components and therefore describes a purely propagating pulse regardless of the specified pulse duration. We then evaluate (S2.4) to obtain

$$\psi_1 = \frac{f^{-s-1}}{k_0 \widetilde{R}}. \tag{S2.8}$$

Equation (S2.8) is a fully analytical solution to (S1.2) containing neither singularities nor backward-propagating components. Taking a partial derivative in time of (S2.8) takes us directly to (S1.1):

$$\psi = \frac{1}{\omega_0}\frac{\partial \psi_1}{\partial t}. \tag{S2.9}$$

Fig. S2.1 summarizes the salient differences between the TD wavepacket and the typical Gaussian beam solution obtained from the Helmholtz equation (S2.2), in the paraxial regime.

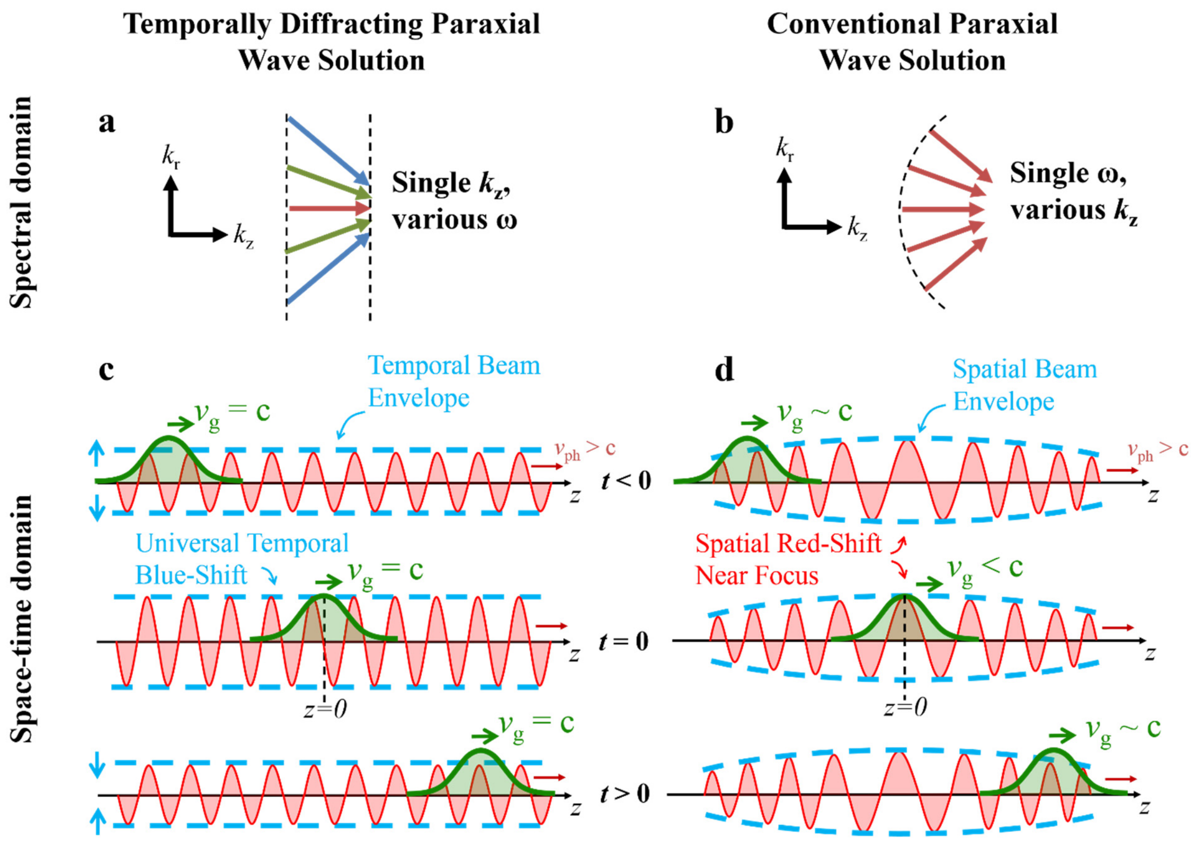

Figure S2.1. An alternative paradigm of formulating electromagnetic pulses, such that the Fourier transform limit does not restrict the shape of the high intensity region at the focus. In the paraxial regime, TD electromagnetic wavepackets are composed of multiple frequencies (a), instead of multiple wavevector component k_z's (b). To obtain finite-energy pulses in the space-time domain, we have to integrate (a) and (b) over multiple k_z and ω respectively. These pulsed versions are shown in (c) and (d), where a spatiotemporal pulse envelope (green) travels along a spatial beam envelope (blue dashes) with the carrier illustrated in red. The overall spatiotemporal pulse envelope is given by multiplying the green and blue portions. The pulse in (d) is associated with a spatial red-shift at the focus as a result of the Gouy phase shift, resulting in a subluminal group velocity near the focus. On the other hand, the TD electromagnetic wavepacket (c) features a beam envelope (and hence a Gouy phase shift) that is a function of time, and an on-axis group velocity v_g that is always exactly the speed of light in free space. The phase velocity v_{ph} is always superluminal in both cases.

3 Spectral domain analysis and fully closed-form spectrum expressions

In this section, we derive and present fully analytical formulas for the spectral domain representations of forward-propagating solution (S1.1). By showing that (S1.1) contains no DC components and has non-zero components only in the positive ω-k_z quadrant of the spectral domain, this analysis also confirms that (S1.1) (which is also Eq. (4) of the main text) is an impeccable description of a forward-propagating pulse.

We define (S1.1) in the spectral domain as

$$\Psi(k_r, k_z, \omega) = \hat{\mathrm{F}}_t \hat{\mathrm{F}}_z \hat{\mathrm{H}}_0 \psi(r, z, t), \tag{S3.1}$$

where Fourier transform operators $\hat{\mathrm{F}}_t, \hat{\mathrm{F}}_z$ and the zeroth-order Hankel transform operator $\hat{\mathrm{H}}_0$ are defined, together with their inverses, as (G, g some azimuthally-symmetric function)

$$\hat{\mathrm{F}}_t g \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g \exp(\mathrm{i}\omega t) \mathrm{d}t, \tag{S3.2a}$$

$$\hat{\mathrm{F}}_t^{-1} G \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G \exp(-\mathrm{i}\omega t) \mathrm{d}\omega, \tag{S3.2b}$$

$$\hat{\mathrm{F}}_z g \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g \exp(-\mathrm{i}k_z z) \mathrm{d}z, \tag{S3.3a}$$

$$\hat{\mathrm{F}}_z^{-1} G \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G \exp(\mathrm{i}k_z z) \mathrm{d}k_z, \tag{S3.3b}$$

$$\hat{\mathrm{H}}_0 g \equiv \int_0^{\infty} g \mathrm{J}_0(k_r r) r \mathrm{d}r, \tag{S3.4a}$$

$$\hat{\mathrm{H}}_0^{-1} G \equiv \int_0^{\infty} G \mathrm{J}_0(k_r r) k_r \mathrm{d}k_r, \tag{S3.4b}$$

where J_0 is the zeroth order Bessel function of the first kind. Due to the dispersion relation in a linear, homogeneous, time-invariant, isotropic medium, Ψ must take the form

$$\begin{aligned}\Psi(k_r,k_z,\omega) &= \Psi_0(k_r,k_z,\omega)\delta\left(k_r^2+k_z^2-\omega^2/c^2\right)\\ &= \Psi_0(k_r,k_z,\omega)\left[\frac{\delta\left(k_r-\sqrt{\omega^2/c^2-k_z^2}\right)}{2\sqrt{\omega^2/c^2-k_z^2}}-\frac{\delta\left(k_r+\sqrt{\omega^2/c^2-k_z^2}\right)}{2\sqrt{\omega^2/c^2-k_z^2}}\right],\end{aligned} \tag{S3.5}$$

where δ is the Dirac delta distribution. Using the expression for Ψ in (S3.5), we note that

$$\begin{aligned}\hat{F}_t\hat{F}_z\psi(r=0,z,t) &= \int_0^\infty \Psi k_r \mathrm{d}k_r\\ &= \frac{1}{2}\Psi_0\left(\sqrt{\omega^2/c^2-k_z^2},k_z,\omega\right),\end{aligned} \tag{S3.6}$$

from which we see that $\Psi_0\left(\sqrt{\omega^2/c^2-k_z^2},k_z,\omega\right)$, <u>the spectral function of (S1.1) on the surface of the light cone</u>, may be calculated as

$$\Psi_0\left(\sqrt{\omega^2/c^2-k_z^2},k_z,\omega\right)=2\hat{F}_t\hat{F}_z\psi(r=0,z,t). \tag{S3.7}$$

Evaluating the right-hand-side of (S3.7) is readily done with standard Fourier transform techniques, yielding the fully analytical expression

$$\Psi_0\left(\sqrt{\omega^2/c^2-k_z^2},k_z,\omega\right)=-2i\sqrt{2\pi}\,\frac{\omega}{k_0^2c^2}\exp\left[-\left(\frac{\omega}{c}-k_z\right)a\right]F(k_z)\theta\left(\frac{\omega}{c}-k_z\right). \tag{S3.8}$$

Equation (S3.8) is 0 when $\omega=0$, showing that it is completely free of DC components. In addition, (S3.8) contains the Heaviside factors $\theta(\omega-k_z c)$ and $\theta(k_z)$, which ensure that it has non-zero components only in the positive ω-k_z quadrant of the spectral domain, on the surface of the light cone. From these observations, we conclude that (S1.1) is an impeccable description of a forward-propagating pulse. We note in passing that these conclusions are also valid for all vector electromagnetic counterparts of (S1.1) since the operations that take the Hertz vectors to the electromagnetic fields (S1.3) are linear and thus do not introduce new spectral components.

A1 Proving the absence of singularities in our exact pulse description

The objective of this section is to prove that the pulse solution in (S1.1), which is also Eq. (4) of the main text, contains no singularities, which makes it the first exact and closed-form expression of a physical electromagnetic pulse. It is sufficient to establish the following statements:

1. $\mathrm{Re}\{\tilde{R}\} \geq a, \forall r,t,a \in \Re, a>0$ and

2. $\mathrm{Re}\{f\} \geq 1, \forall r,z,t,a,s \in \Re, a,s>0$,

where all variables are as defined in Sections 1 and 2. To begin with, note that

$$\tilde{R} \equiv \sqrt{r^2 + (\mathrm{i}ct + a)^2} = \alpha + \mathrm{i}\beta, \tag{SA1.1}$$

where $\alpha, \beta \in \Re$. We solve (SA1.1) to obtain

$$\alpha = \sqrt{\frac{r^2 - c^2t^2 + a^2}{2} + \sqrt{\left(\frac{r^2 - c^2t^2 + a^2}{2}\right)^2 + a^2c^2t^2}}\ , \tag{SA1.2a}$$

$$\beta = \frac{act}{\alpha}. \tag{SA1.2b}$$

In arriving at solutions (SA1.2), we choose the branches for the square-roots such that $\alpha \in \Re$ and $\alpha > 0$, the latter being necessary for a physically valid solution given the form of (S2.6) and the fact that we build our solution (S1.1) from components with $k_z \geq 0$. For convenience, we define

$$h \equiv \sqrt{\left(\frac{r^2 - c^2t^2 + a^2}{2}\right)^2 + a^2c^2t^2}\ . \tag{A1.3}$$

Now consider

$$h' \equiv -\frac{r^2 - c^2t^2 + a^2}{2} + a^2 . \tag{SA1.4}$$

Note that

$$h'^2 = \left(\frac{r^2 - c^2t^2 + a^2}{2}\right)^2 + a^2c^2t^2 - a^2r^2 \le h^2 , \tag{SA1.5}$$

from which it follows that $h' \le h$. This implies that

$$\begin{aligned} \alpha &= \sqrt{\frac{r^2 - c^2t^2 + a^2}{2} + h} \\ &\ge \sqrt{\frac{r^2 - c^2t^2 + a^2}{2} + h'} = a \end{aligned} , \tag{SA1.6}$$

so $\alpha \ge a$, showing that Statement 1 is correct. Statement 2 immediately follows from this since

$$\mathrm{Re}\{f\} = 1 + \frac{k_0}{s}(\alpha - a). \tag{SA1.7}$$